From Cascades to Multifractal Processes

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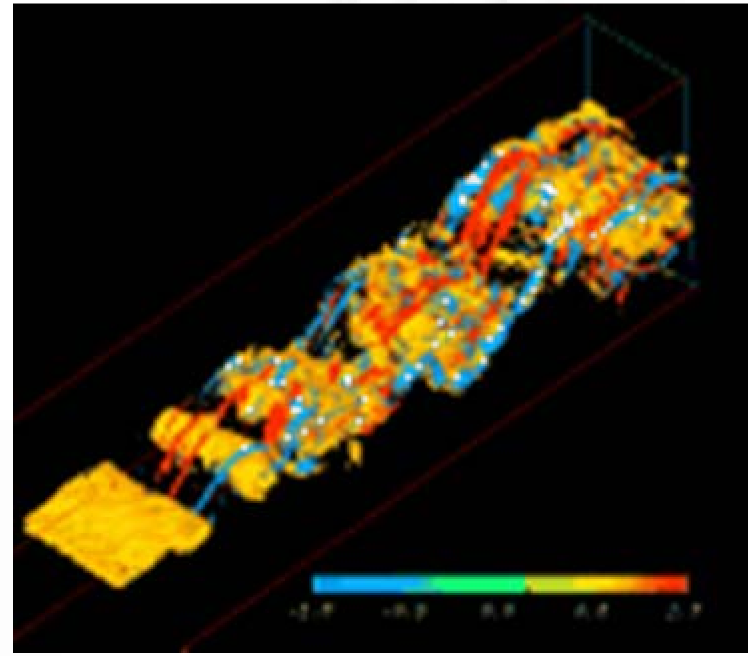
Reading on this talk

- www.stat.rice.edu/~riedi
- This talk
- Intro for the "untouched mind"
  - Explicit computations on Binomial
- Monograph on "Multifractal processes"
  - Multifractal formalism (proofs, references)
  - Multifractal subordination (warping)
- Papers, links

Why Cascades



Courtesy P. Chainais

Turbulence: models wanted

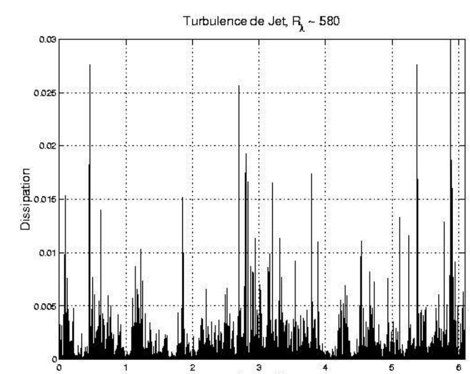
- Kolmogorov 1941 :

$< [v(x+r)-v(x)]^q > \sim r^{q/3}$

- Kolmogorov 1962 :

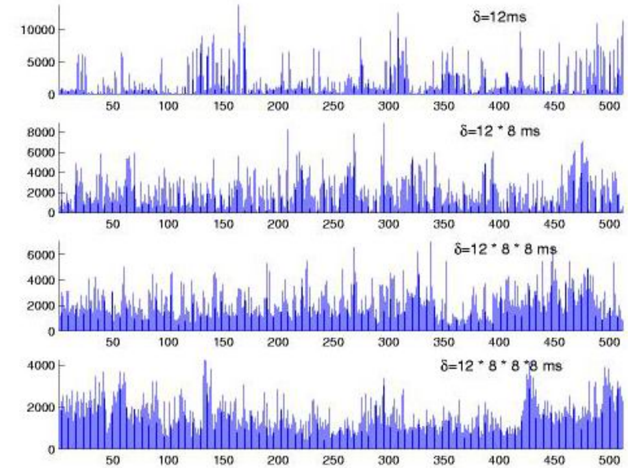
$< [v(x+r)-v(x)]^q > \sim r^{H(q)}$

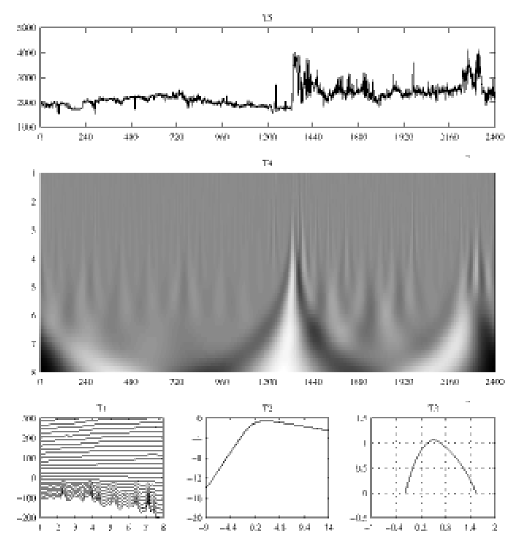
- …and beyond

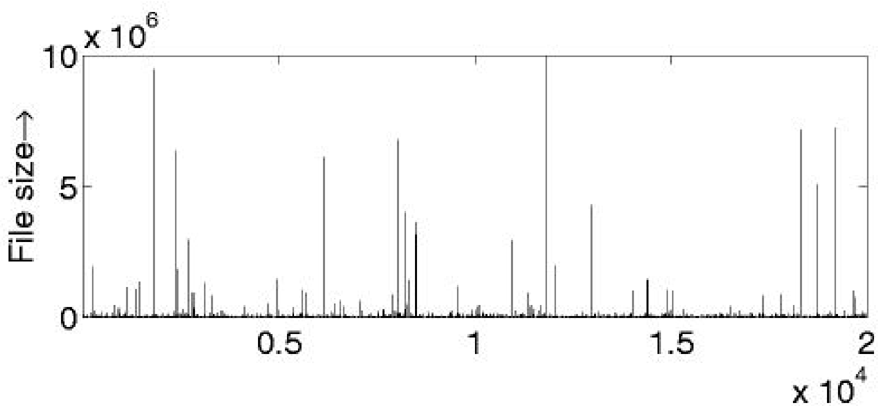


Measured Data

- Networks
- Geophysics
- WWW
- Stock Markets







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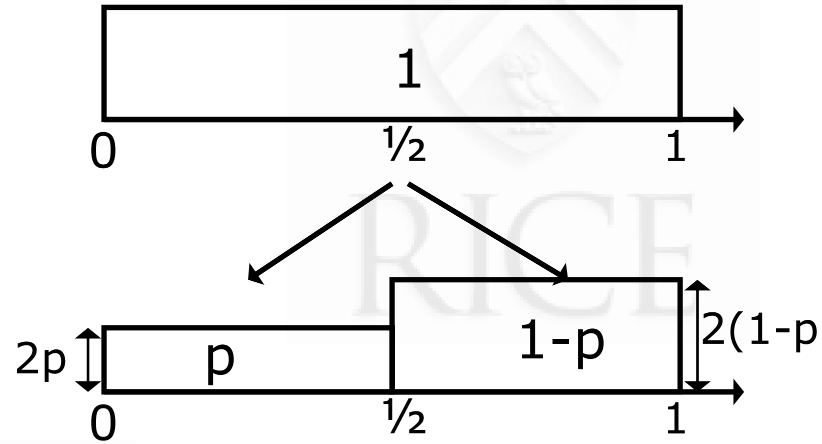
Multifractal Analysis

Toy Example

The Toy: Binomial Cascade

- Start with unit mass
- Redistribute uniformly
  portion p <½ to the left
  portion 1-p to the right



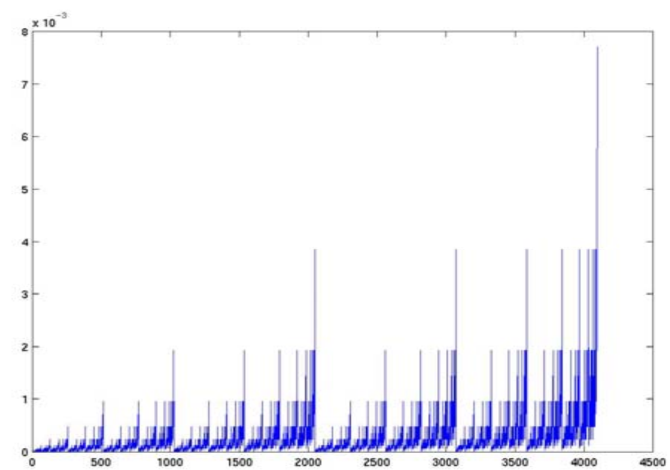
Mass re-distribution

- Iterate
- Converges to measure μ

$$t = \sum_{k=1}^{\infty} \epsilon_k / 2^k \quad \text{with } \epsilon_k = 0, 1$$

$$I(\epsilon_1 \ldots \epsilon_n) := [\sum_{k=1}^{n} \epsilon_k/2^k, \sum_{k=1}^{n} \epsilon_k/2^k + 1/2^n))$$

$$l_n(t) := \#\{k \leq n : \epsilon_k = 1\} = \sum_{k=1}^{n} \epsilon_k$$



$$\mu(I(\epsilon_1 \ldots \epsilon_n)) = p^{n-l_n(t)}(1-p)^{l_n(t)}$$

Multifractal Spectrum

- Oscillate ~ $|t|^{\alpha}$ → local strength α

$$\alpha(t) := \liminf_n \alpha_n(t)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

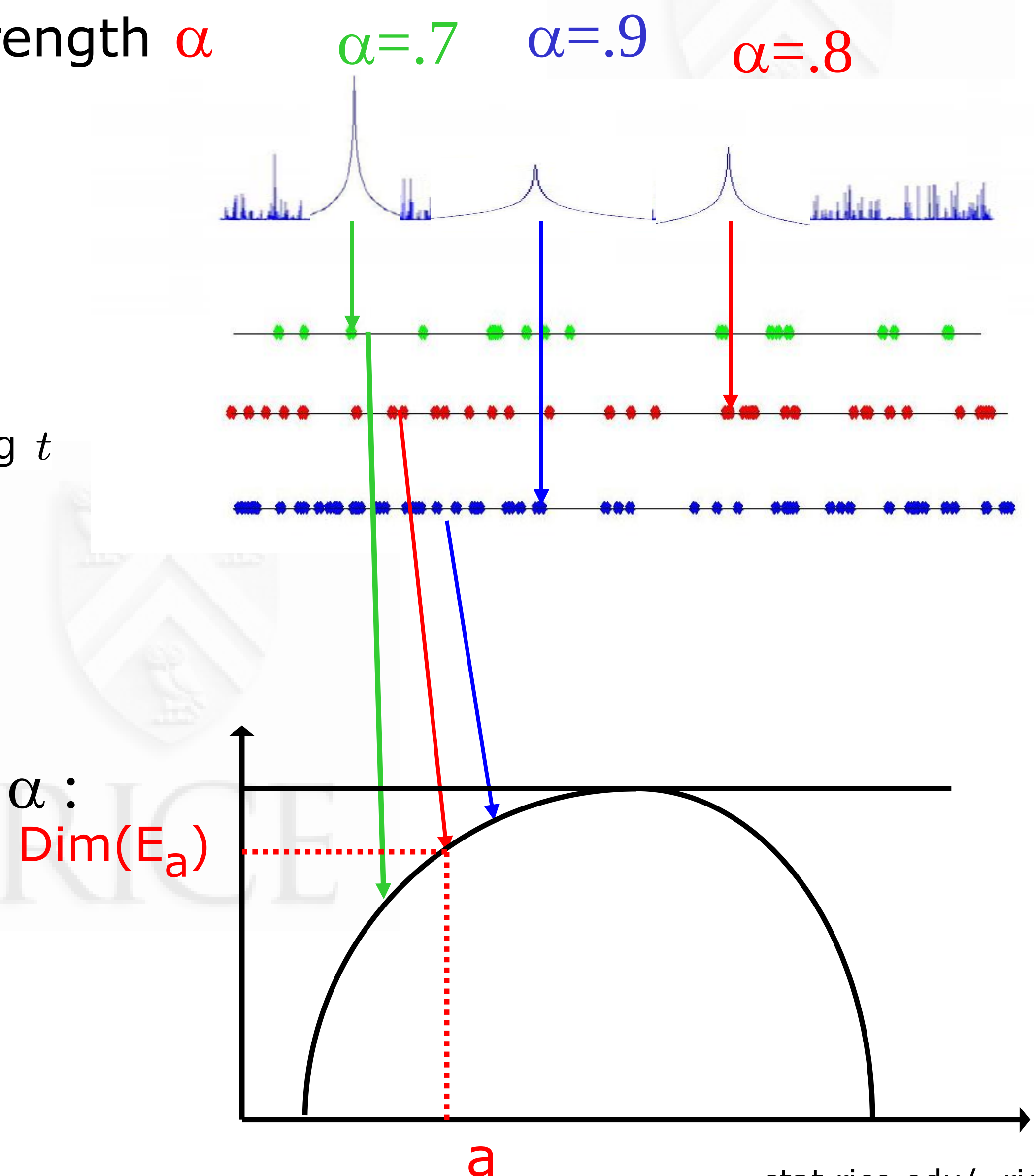
$I_n(t)$: dyadic interval containing t

$\Delta I_n(t)$: oscillation indicator
total increment over I_n,
max increment in I_n,
wavelet coefficients,...

- Collect points t with same α :

$$E_a := \{t \, : \, \alpha(t) = a\}$$

- Dim(E_a): Spectrum
→prelevance of α



Recall

$$\alpha(t) := \liminf_n \alpha_n(t)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$l_n(t) = \#\{k \le n : \epsilon_k = 1\}$$

Binomial

We take dyadic partition:

$$I_n(t) = I(\epsilon_1 \ldots \epsilon_n) := [\sum_{k=1}^{n} \epsilon_k/2^k, \sum_{k=1}^{n} \epsilon_k/2^k + 1/2^n))$$

$$\begin{aligned}\Delta I_n(t) &= \mu(I_n(t)) \\ &= p^{l_n(t)}(1-p)^{n-l_n(t)}\end{aligned}$$

$$\alpha_n(t) = -\frac{n - l_n(t)}{n}\log_2(p) - \frac{l_n(t)}{n}\log_2(1-p)$$

Range of exponents:

$$t = 0: \ l_n = 0, \ \alpha_n \to \quad a_\infty := -\log_2(p) \qquad < 1$$
$$t = 1: \ l_n = n, \ \alpha_n \to a_{-\infty} := -\log_2(1-p) > 1$$

"Typical" exponents

Recall

$$\alpha(t) := \liminf_n \alpha_n(t)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$l_n(t) = \#\{k \le n : \epsilon_k = 1\}$$

t=0, t=1 seem "atypical".

Intuition: for a "typical" t:

$$l_n(t) \simeq n/2$$

Rigorously: Law of Large Numbers

- Binary digits ϵ_k are independent, $P[\epsilon_k=0]= P[\epsilon_k=1]= ½$:
- t is uniformly distributed (i.e., with Lebesgue measure $\mathcal{L}$)
- $$\frac{l_n(t)}{n} = \frac{1}{n}\sum_{k=1}^{n} \epsilon_k \to \mathbb{E}_{\mathcal{L}}[\epsilon] = 1/2$$
- "Typical" exponent:

$$\boxed{\begin{aligned} \alpha_n(t) &= -\frac{n-l_n(t)}{n}\log_2(p) - \frac{l_n(t)}{n}\log_2(1-p) \\ &\to a_0 := -\frac{1}{2}\log_2(p) - \frac{1}{2}\log_2(1-p) \quad > 1 \end{aligned}}$$

A first point on the Spectrum

Conclusion:

- $$\mathcal{L}(E_{a_0}) > 0$$

- Mass Distribution Principle
  (Lebesgue measure $\mathcal{L}$ is 1-dim Hausdorff measure)

$$\boxed{\dim E_{a_0} = 1}$$

"Where" and "how many" are the other exponents?

- Choose digits "unfairly", e.g., prefer 1 over 0.

Other exponents

Recall

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$l_n(t) = \#\{k \le n : \epsilon_k = 1\}$$

The measure μ prefers 1 over 0 (ratio 1-p to p).
Intuitive:

$$l_n(t) \simeq n(1-p)$$

Rigorously: Law of Large Numbers using μ

- Binary digits ϵ are independent, $P[\epsilon_k=0]=p$, $P[\epsilon_k=1]= 1-p$:
- t is distributed according to μ
- $$\frac{l_n(t)}{n} = \frac{1}{n}\sum_{k=1}^{n} \epsilon_k \to \mathbb{E}_\mu[\epsilon] = 1-p$$
- μ-typical exponent

$$\boxed{\begin{aligned} \alpha_n(t) &= -\frac{n-l_n(t)}{n}\log_2(p) - \frac{l_n(t)}{n}\log_2(1-p) \\ &\to a_1 := -p\log_2(p) - (1-p)\log_2(1-p) \quad < 1 \end{aligned}}$$

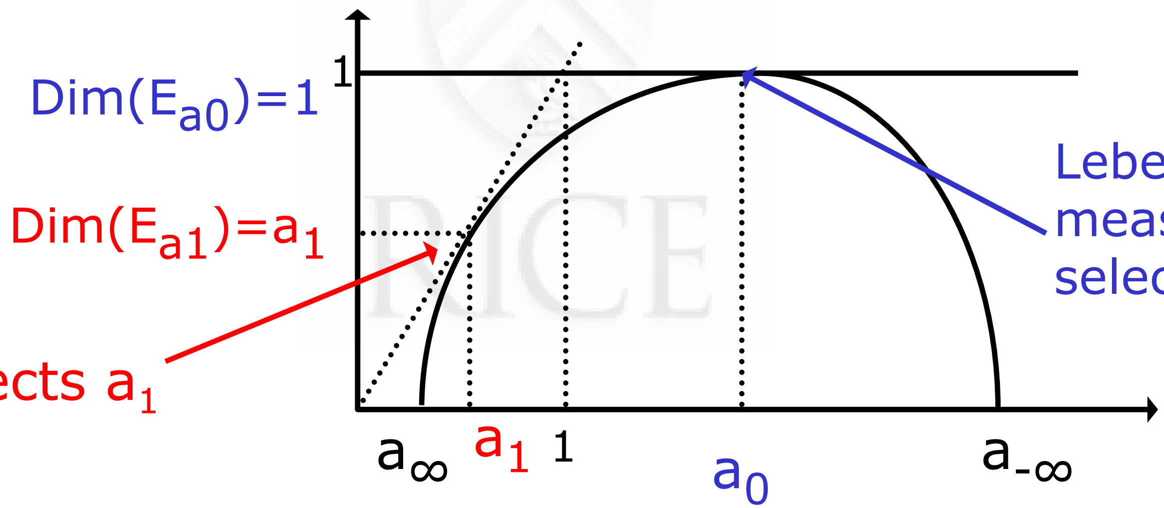
A second point on the Spectrum

Conclusion:

- $\mu(E_{a_1}) > 0$
- Mass Distribution Principle → $\boxed{\dim E_{a_1} \geq a_1}$

(Hausdorff dimension of μ? It is $a_1<1$!)

$$\boxed{\alpha_n(t) \;=\; \frac{\log \mu(I_n(t))}{\log |I_n(t)|} \to a_1}$$



- All exponents: Inspiration from Large Deviation Theory

Large Deviations

and the
Multifractal Formalism

Box Spectrum

Recall

$$I_n(t) = I(\epsilon_1 \dots \epsilon_n)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$E_a := \{t : \alpha(t) = a\}$$

- Notation:

$$N_n(a, \delta) := \#\{(\epsilon_1 \dots \epsilon_n) : a - \delta \leq \alpha_n(\epsilon_1 \dots \epsilon_n) < a + \delta\}.$$

$$f(a) := \lim_{\delta \downarrow 0} \limsup_{n \to \infty} \frac{1}{n} \log_2 N_n(a, \delta)$$

- Thm: we always have

$$\dim E_a \leq f(a)$$

Proof

Fix a. To prove the first part of the lemma consider an arbitrary $\gamma > f(a)$, and choose $\eta > 0$ such that $\gamma > f(a) + 2\eta$. Then, there is $\varepsilon > 0$ and integer m_0 such that

$$N_n(a, \varepsilon) \leq 2^{n(f(a)+\eta)}$$

for all $n > m_0$. Let us define $J(m) := \cup\{k_n : n \geq m \text{ and } a - \varepsilon \leq \alpha_n^k \leq a + \varepsilon\}$. Then, for any m the intervals I_n^k with $k_n \in J(m)$ form a cover of E_a. These intervals are of length less than 2^{-m}. Moreover, for any $m > m_0$ we have

$$\begin{aligned} \sum_{k_n \in J(m)} |I_n^k|^\gamma &= \sum_{n \geq m} N_n(a, \varepsilon) \cdot 2^{-n\gamma} \\ &\leq \sum_{n \geq m} 2^{-n(\gamma - f(a) - \eta)} \leq \sum_{n \geq m} 2^{-n\eta} \end{aligned}$$

tends to zero with $m \to \infty$. We conclude that the γ-dimensional Hausdorff measure of E_a is zero, hence, $\dim E_a \leq \gamma$. Letting $\gamma \to f(a)$ completes the proof.

[www.stat.rice.edu/~riedi]

- Beware the folklore: f(a) is NOT the box-dim of E_a

Legendre spectrum

Recall

$$I_n(t) = I(\epsilon_1 \dots \epsilon_n)$$

$$\alpha_n(t) = \frac{\log \Delta I_n(t)}{\log |I_n(t)|}$$

$$E_a := \{t : \alpha(t) = a\}$$

- Notation: partition sum and function

$$S_n(q) := \sum_{\epsilon_1 \dots \epsilon_n} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q = \sum_{\epsilon_1 \dots \epsilon_n} |2^n|^{q\alpha_n(\epsilon_1 \dots \epsilon_n)}.$$

$$\tau(q) := \liminf_{n \to \infty} -\frac{1}{n} \log_2 S_n(q)$$

- Thm: we always have

$$f(a) \le \tau^*(a) := \inf_q (qa - \tau(q))$$

Proof

$$\sum_{\epsilon_1 \dots \epsilon_n} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q \ge \sum_{\alpha_n(\epsilon_1 \dots \epsilon_n) \in [a-\delta, a+\delta]} |\Delta I_n(\epsilon_1 \dots \epsilon_n)|^q \ge N_n(a,\delta) 2^{-n(qa+|q|\delta)}$$

$$\ge 2^{-n(qa - f(a) + \delta' + |q|\delta)}$$

Legendre spectrum

- Thm: provided $\alpha_n(t)$ are bounded we have

$$f(a) = \tau^*(a) \qquad \text{for } a = \tau'(q).$$

- Proof idea: steepest ascent (large deviations)

$$\begin{aligned}
\sum_{\epsilon_1 \ldots \epsilon_n)} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q &\leq \sum_{l=1}^{m} \sum_{\alpha_n(\epsilon_1 \ldots \epsilon_n) \in [l\delta - \delta, l\delta + \delta]} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q \\
&\leq \sum_{l=1}^{m} N_n(l\delta, \delta) 2^{-n(ql\delta - |q|\delta)} \\
&\leq \sum_{l=1}^{m} 2^{-n(ql\delta - f(l\delta) - \delta' - |q|\delta)} \leq m 2^{-n(\underline{\inf_a(qa - f(a)} - \delta' - |q|\delta)}
\end{aligned}$$

  - Thus: $\underline{\tau(q) = f^*(q)} = \inf_a (qa - f(a)) \quad \text{for all } q.$
  - τ is concave, non-decreasing, differentiable with exceptions
  - Recover f=f** at a=τ′(q) using lower semi-continuity

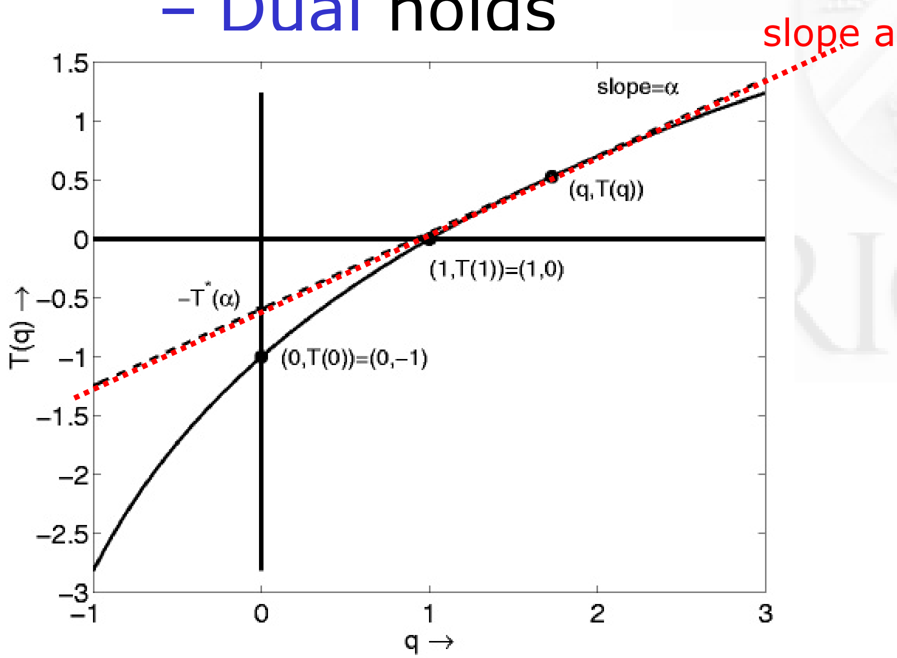
Legendre transform 101

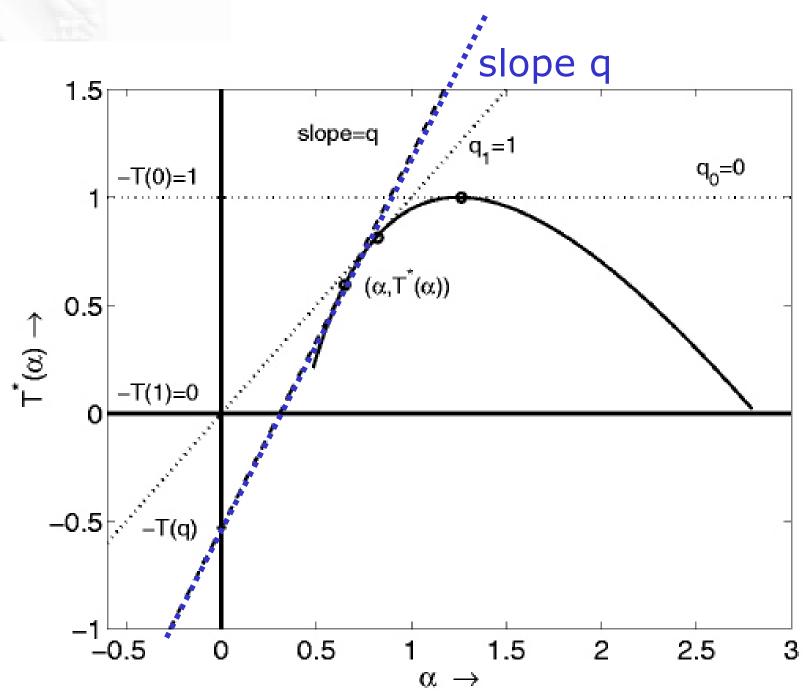
- Elementary calculus:

$$\tau^*(a) := \inf_q (qa - \tau(q)) = \bar{q}a - \tau(\bar{q})$$

where $\bar{q}$ is defined by $a = \tau'(\bar{q})$

  - Tangent of slope a to τ(q)
  - Intersection with ordinate yields $-\tau^*(a)$
  - Dual holds





Binomial Spectrum

continued

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Partition function of the Binomial

$$\begin{aligned} S_n(q) &= \sum_{\epsilon_1\ldots\epsilon_n} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q \\ &= \sum_{\epsilon_1\ldots\epsilon_n} [p^{n-l_n(\epsilon_1\ldots\epsilon_n)}(1-p)^{l_n(\epsilon_1\ldots\epsilon_n)}]^q \\ &= \sum_{l=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} [p^{n-l}(1-p)^l]^q \\ &= [p^q + (1-p)^q]^n. \end{aligned}$$

- (Upper) envelop of dim(E_a):

$$\begin{aligned} \tau(q) &= \liminf_{n\to\infty} -\frac{1}{n}\log_2 S_n(q) \\ &= -\log_2[p^q + (1-p)^q] \end{aligned}$$

Insight from Large Deviations

- From steepest ascent:

$$\begin{aligned} S_n(q) &= \sum_{\epsilon_1\ldots\epsilon_n} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q \simeq 2^{-n(\inf_a(qa-f(a))} \\ &= 2^{-n(q\overline{a}-f(\overline{a}))} \simeq \sum_{\alpha_n(\epsilon_1\ldots\epsilon_n)\simeq a} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q \end{aligned}$$

  - Dominant terms in $S_n(q)$, for fixed q, are the ones with

$$\alpha_n(\epsilon_1 \ldots \epsilon_n) = \frac{\log \Delta I_n}{\log |I_n|} \simeq \overline{a} = \tau'(q)$$

    - …and vice versa: these terms contribute such that

$$S_n(q) \simeq 2^{-n\tau(q)} = (p^q + (1-p)^q)^n$$

For the Binomial these correspond
to mass re-distribution in ratio p^q to $(1-p)^q$

Locating the exponents

Fix q.

Consider the measure μ_q defined as μ but with mass ratio p^q to $(1-p)^q$. Intuitively, we have then:

$$l_n(t) = \#\{k \le n : \epsilon_k = 1\} \simeq n \frac{(1-p)^q}{p^q + (1-p)^q} = n(1-p)^q 2^{\tau(q)}$$

Rigorously: Law of Large Numbers using μ_q

- Binary digits ϵ: indep, $P[\epsilon_k=0]=p^q 2^{\tau(q)}$, $P[\epsilon_k=1]= (1-p)^q 2^{\tau(q)}$
- t is distributed according to μ_q
- $$\frac{l_n(t)}{n} = \frac{1}{n}\sum_{k=1}^{n} \epsilon_k \to \mathbb{E}_{\mu_q}[\epsilon] = (1-p)^q 2^{\tau(q)}$$
- μ-typical exponent

$$\boxed{\begin{aligned} \alpha_n(t) &= -\frac{n - l_n(t)}{n}\log_2(p) - \frac{l_n(t)}{n}\log_2(1-p) \\ &\to a_q := -p^q 2^{\tau(q)}\log_2(p) - (1-p)^q 2^{\tau(q)}\log_2(1-p) \end{aligned}}$$

Completing the Spectrum

Recall

$$\begin{aligned}\tau(q) &= -\log_2[p^q + (1-p)^q] \\ a_q &= -p^q 2^{\tau(q)} \log_2(p) - (1-p)^q 2^{\tau(q)} \log_2(1-p)\end{aligned}$$

Conclusion:

- $\mu_q(E_{a_q}) > 0$
- $a_q = \tau'(q)$

- Hausdorff dimension of μ_q:

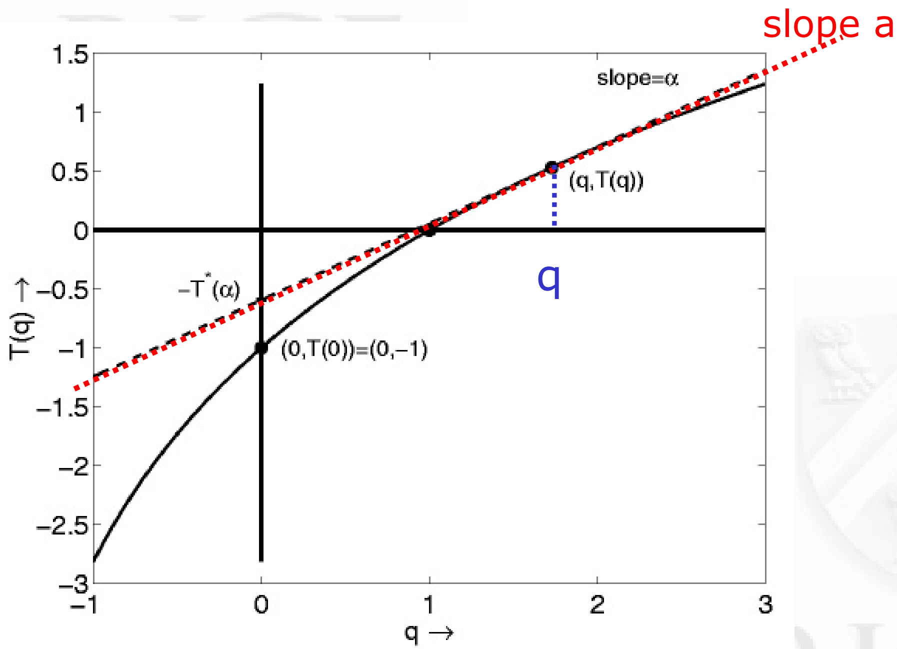
$$\begin{aligned}\frac{\log \mu_q(I_n(t))}{\log |I_n(t)|} &= -\frac{n - l_n(t)}{n} \log_2[p^q 2^{\tau(q)}] - \frac{l_n(t)}{n} \log_2[(1-p)^q 2^{\tau(q)}] \\ &= -\tau(q) + q\alpha_n(t) \\ &\to qa_q - \tau(q) = \tau^*(a_q)\end{aligned}$$

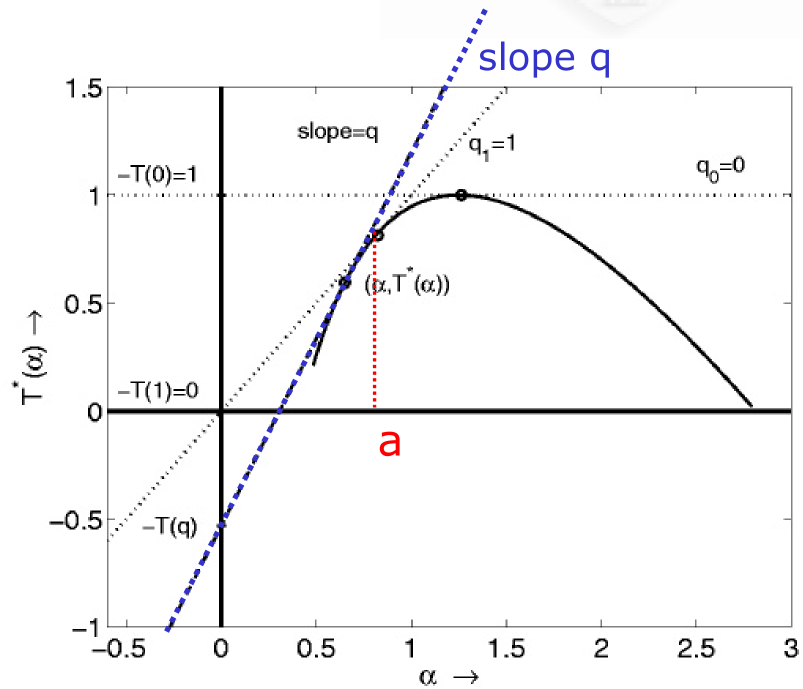
- Mass Distribution Principle

$$\boxed{\dim E_{a_q} \geq \tau^*(a_q)}$$

Lessons

Binomial cascade: $\dim E_a = f(a) = \tau^*(a)$





- Points with exponent $\log\mu(I(t))/\log|I(t)| \sim a=\tau'(q)$
  - Are concentrated on the support of μ_q
  - Dominate the partition sum $S_n(q)$
- Partition function allows to bound/estimate $\dim(E_a)$

Random Cascades

A further multifractal envelop
Convergence and Degeneracy

Multifractal Spectra and Randomness

- $\Delta I_n(t)$: oscillation indicator for process or measure
- Pathwise

$$\dim E_a \le f(a) \le \tau^*(a)$$

Recall

$$E_a = \{t : \alpha(t) = a\}$$

$$N_n(a,\varepsilon) \simeq 2^{nf(a)}$$

$$S_n(q) \simeq 2^{-n\tau(q)}$$

- $S_n(q)$ is q-th moment estimator.
- Replace by true moment:

Recall

$$S_n(q) = \sum_{\epsilon_1 \ldots \epsilon_n} |\Delta I_n(\epsilon_1 \ldots \epsilon_n)|^q$$

$$T(q) := \liminf_{n\to\infty} -\frac{1}{n} \log_2 \mathbb{E} S_n(q)$$

- …analytically easier to handle and often sufficient
- T(q) is concave like $\tau(q)$, but NOT always increasing

Pathwise and deterministic envelop

- Lemma: With probability one for all q with $T(q) < \infty$.

$$\tau(q,\omega) \geq T(q)$$

[Proof: www.stat.rice.edu/~riedi]

- Cor: $$\tau^*(a,\omega) \leq T^*(a)$$

$$\boxed{\mathbb{E}[\log(X)] \leq \log \mathbb{E}[X]}$$

- Weaker result from Chebichev inequality:

$$\mathbb{E}[\tau(q,\omega)] \geq T(q)$$

Quenched Average Annealed Average

- Material science: free energy is "self-averaging" iff quenched and annealed averages are equal.

Multifractal Envelops

- Almost surely, for all a:

Recall at $a=\tau'(q)$
$f(a)=\tau^*(a)$

$$\dim E_a \leq f(a) \leq \tau^*(a) \leq T^*(a)$$

- Holds always provided use same ΔI_n in all spectra
- Choice of scales I_n
  - I_n is here dyadic, could be any sub-exponential
  - This could affect/change f, τ and/or T due to boundary effects
  - Robust: ΔI_n = oscillation in I_n and its neighbor intervals
- Choice of oscillation indicator ΔI_n
  - For true Hoelder regularity ΔI_n = max increment "around" I_n
  - ΔI_n = Wavelet coefficient: only a proxy to Hoelder regularity!
  - For measures supported on [0,1]: $\Delta I_n = \mu(I_n)$ gives Hoelder!

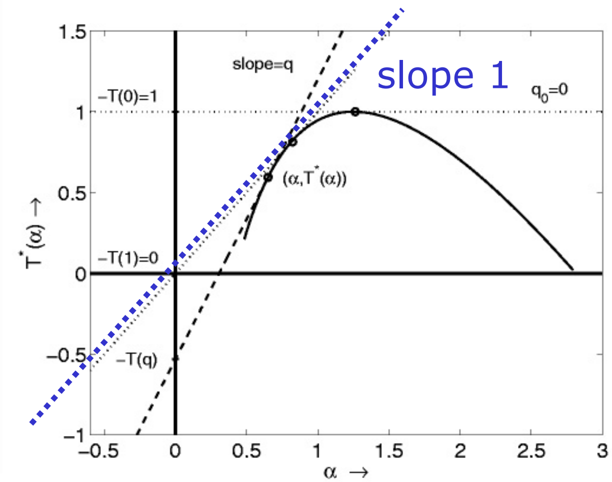
Multifractal Envelops

Recall at $a = \tau'(q)$
$f(a) = \tau^*(a)$

- Almost surely, for all a:

$$\dim E_a \le f(a) \le \tau^*(a) \le T^*(a)$$

- Special feature:
  - If a property of "bounded total variation" holds then the spectrum f touches the bi-sector:



If $\sum_{\epsilon_1\ldots\epsilon_n} \Delta I_n(\epsilon_1 \ldots \epsilon_n) \le C$ for all n
then $\tau(1) = 0$.

Multifractal Envelops

Recall at $a = \tau'(q)$
$f(a) = \tau^*(a)$

- Almost surely, for all a:

$$\dim E_a \leq f(a) \leq \tau^*(a) \leq T^*(a)$$

- Terminology:
  - Multifractal formalism "holds" if
    - $\dim(E_a)=f(a)=\tau^*(a)$

    with your preferred oscillation indicators ΔI_n,
    e.g., Holder exponent in E_a, wavelet decay in f(a).
    [First step: show T is same for Holder and wavelets.]

  - Falconcer: "A concise definition of a multifractal tends to be avoided."
  - Others: "An object is multifractal if the formalism holds for it."
  - Others: "An object is multifractal if it has more than one singularity exponent". (not mono-fractal)

Multifractals and classical regularity

Besov spaces

- For oscillation indicator from wavelets:

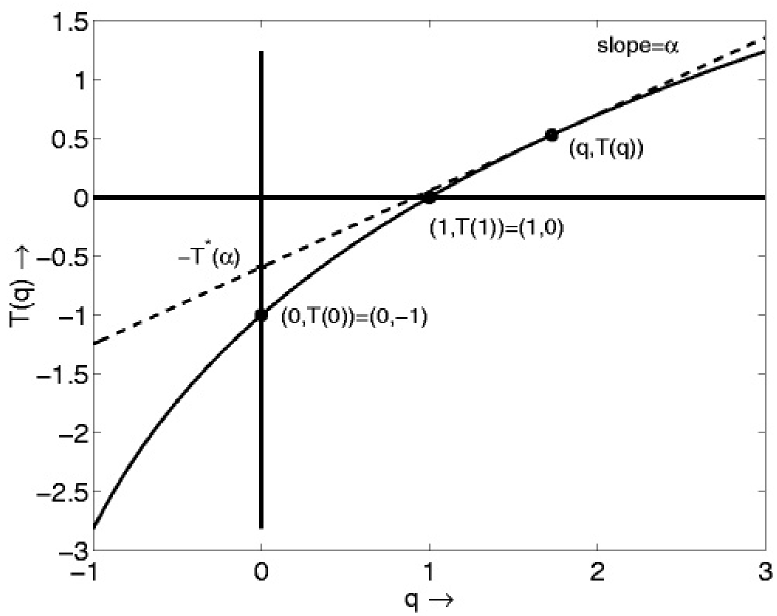
$$\text{sup}\{s \,:\, Y \in B_v^s(L^u)\} = \frac{\tau(u)+1}{u}$$

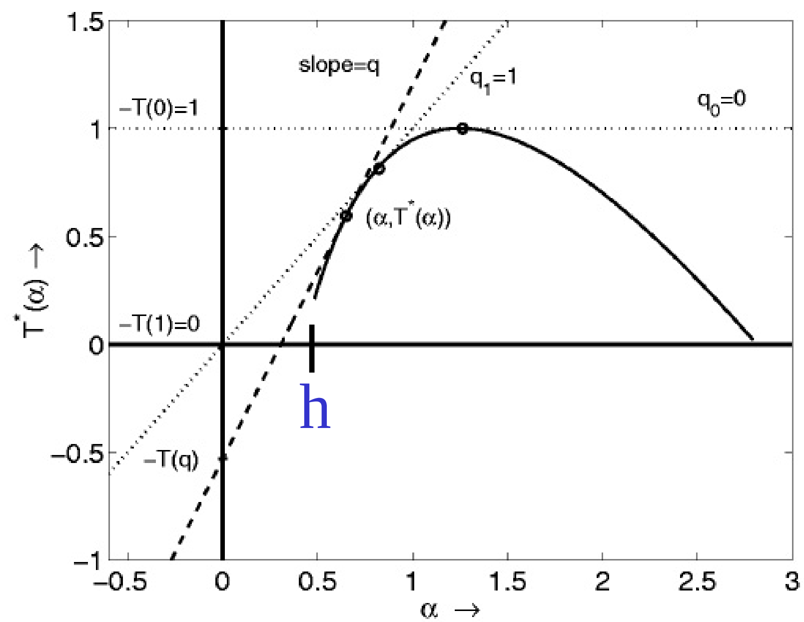
- Proof: use wavelet coefficients $C_{j,k} = \Delta I_j(k2^j)$ and equivalent Besov norm

$$\left(\sum_k |D_{0,0}|^v\right)^{1/v} + \left(\sum_{j \geq J_0}\left(\sum_k 2^{jsu} 2^{-j}\left|2^{j/2} C_{j,k}\right|^u\right)^{v/u}\right)^{1/v}.$$

Kolmogorov

- Thm [Kolmogorov]:
  - If E[| A(s)-A(t) |^b] < C | s-t |$^{1+d}$ then almost all paths of A are of (global) Holder-continuity for all h < d/b,
  - i.e., for all h < T(q)/q .
- The best such h is min(a : T*(a)>0).
  - T(q)/q = slope of tangent through the origin.





Binomial Spectrum

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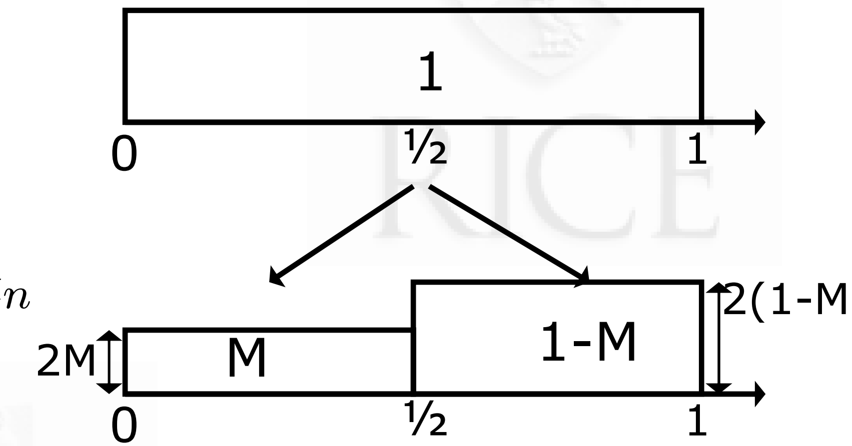
Binomial with Random Multipliers

- Random re-distribution
- Multipliers Independent
  between scales

$$\mu_n(I(\epsilon_1 \ldots \epsilon_n)) = M_{\epsilon_1} \ldots M_{\epsilon_1 \ldots \epsilon_n}$$

- Conservative:

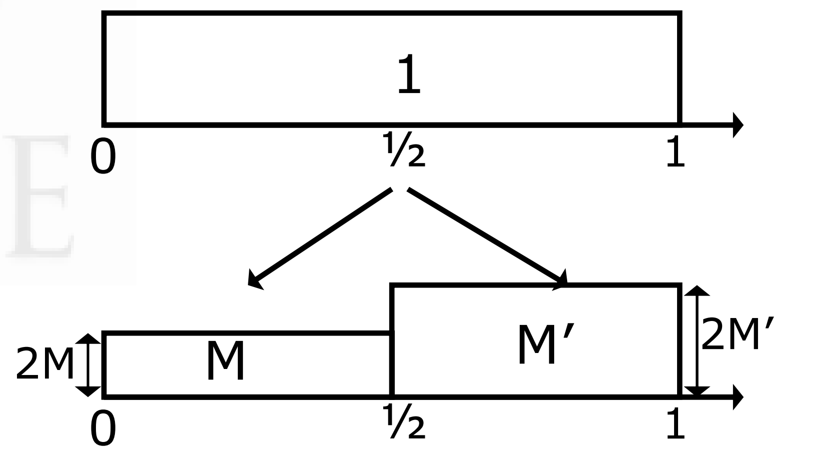
$$M_{\epsilon_1 \ldots \epsilon_n 0} + M_{\epsilon_1 \ldots \epsilon_n 1} = 1$$



Conservative re-distribution

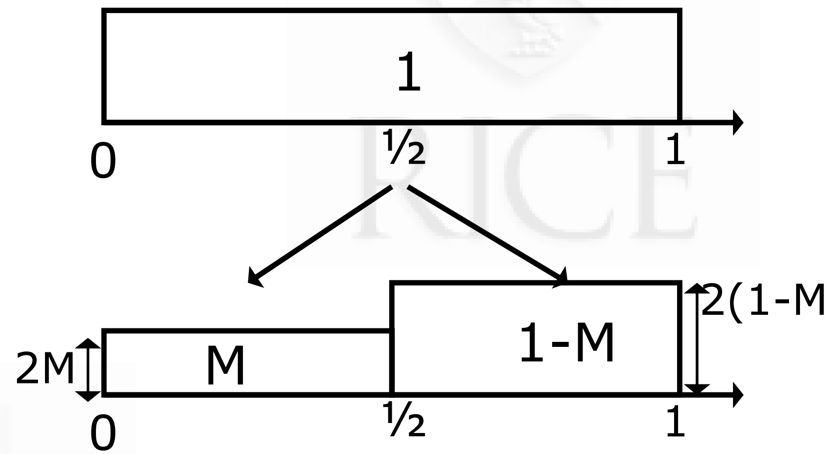
- Conservation is too restrictive
  for stationarity!
- “Martingale de Mandelbrot”:

$$\mathbb{E}[M_{\epsilon_1 \ldots \epsilon_n 0} + M_{\epsilon_1 \ldots \epsilon_n 1}] = 1$$



Independent re-distribution

Convergence of Random Binomial



Conservative re-distribution

- Conservative:
  - $M_{\epsilon_1 \ldots \epsilon_n 0} + M_{\epsilon_1 \ldots \epsilon_n 1} = 1$
  - For all m>n

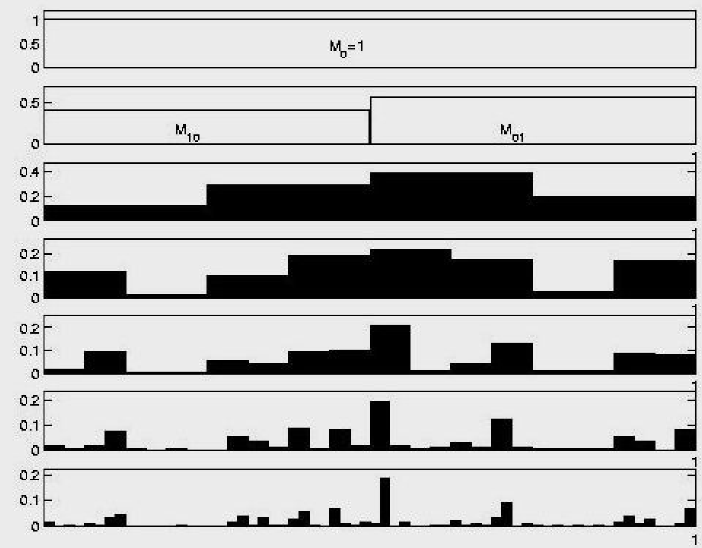
    $$\mu_m(I(\epsilon_1 \ldots \epsilon_n)) = M_{\epsilon_1} \ldots M_{\epsilon_1 \ldots \epsilon_n}$$

  - Thus converges to

$$\mu(I(\epsilon_1 \ldots \epsilon_n)) = M_{\epsilon_1} \ldots M_{\epsilon_1 \ldots \epsilon_n}$$

Convergence of Random Binomial

- "Martingale de Mandelbrot":
  - A price to pay towards stationarity
  - $\mathbb{E}[M_{\epsilon_1\ldots\epsilon_n 0} + M_{\epsilon_1\ldots\epsilon_n 1}] = 1$



  - Martingale: For all m>n

$$\mathbb{E}[\mu_m(I(\epsilon_1 \ldots \epsilon_n))|\mathcal{F}_n] = M_{\epsilon_1} \ldots M_{\epsilon_1\ldots\epsilon_n} = \mu_n(I(\epsilon_1 \ldots \epsilon_n))$$

  - Thus converges almost surely (but may degenerate)
  - We have

$$\mathbb{E}[\mu(I(\epsilon_1 \ldots \epsilon_n))|\mathcal{F}_n] = M_{\epsilon_1} \ldots M_{\epsilon_1\ldots\epsilon_n}$$

Envelope for Random Binomial

- By independence of multipliers
  - Martingale of Mandelbrot:

$$\mathbb{E}[S_n(q)] \;=\; \sum_{\epsilon_1\dots\epsilon_n} \mathbb{E}|\Delta I_n(\epsilon_1\dots\epsilon_n)|^q = \sum_{\varepsilon_1\dots\varepsilon_n} \mathbb{E}|M_{\epsilon_1}\dots M_{\epsilon_1\dots\epsilon_n}|^q = 2^n\mathbb{E}[M^q]^n.$$

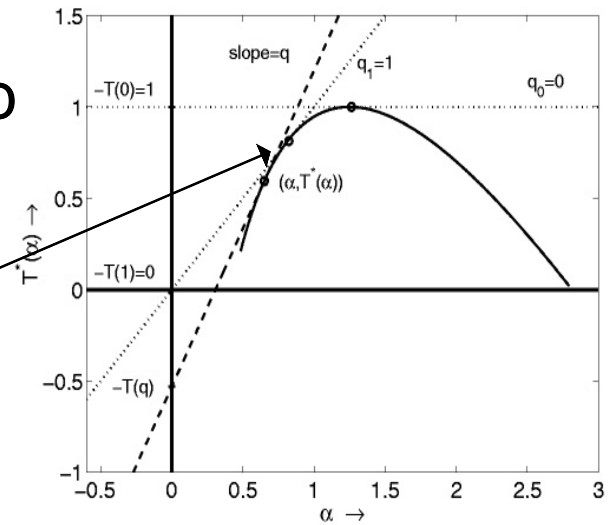
$$T(q) = -1 - \log_2 \mathbb{E}[M^q]$$

  - Conservative: similar

$$\begin{aligned}\mathbb{E}[S_n(q)] &= \sum_{\epsilon_1\dots\epsilon_n} \mathbb{E}[M^q]^{n-l_n(\epsilon_1\dots\epsilon_n)}\mathbb{E}[(1-M)^q]^{l_n(\epsilon_1\dots\epsilon_n)} \\ &= (\mathbb{E}[M^q] + \mathbb{E}[(1-M)^q])^n \\ &= (2\mathbb{E}[M^q])^n\,.\end{aligned}$$

Kahane-Peyriere theory for the Martingale of Mandelbrot

- Martingale “degenerates”
  - iff $\mu([0,1])=0$ almost surely zero
  - iff E $\mu([0,1])=0$
  - iff $T'(1)<=0$



- Intuition:
  - $T'(1)= a_1$ = dimension of the carrier of μ.
  - If $T'(1)>0$ then
    - $\exists q>1$ with $T(q)>0$
    - μ converges in L_q
    - $\mathbb{E}[\mu([0,1])] = \lim_n \mathbb{E}[\mu_n([0,1])] = 1$

Multifractal formalism holds

- Thm for random binomial [Barral, Arbeiter-Patschke, Falconer]:
  - Set $\Delta I_n = \mu(I_n)$.
  - Assume M has a finite moment of some negative order
  - Then, with probability 1: for all a such that $T^*(a)>0$

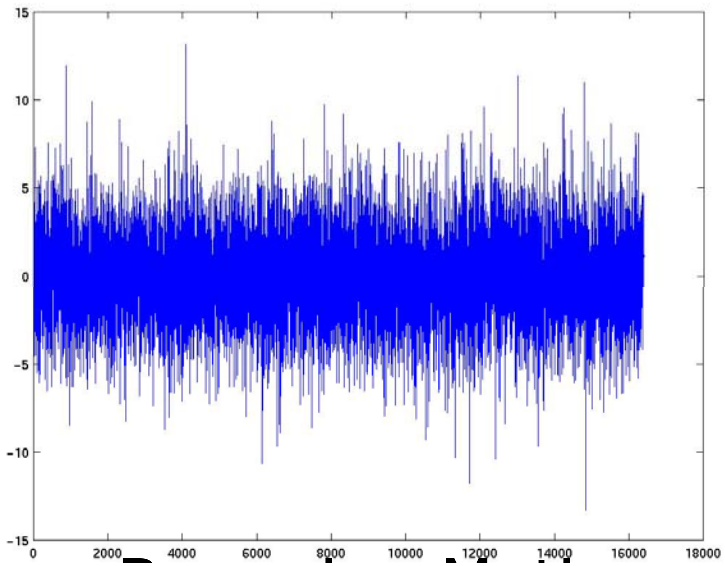
$$\dim E_a = f(a) = \tau^*(a) = T^*(a)$$

- Note:
  - $T^*(a)>0$ means $a=T'(q)$ with q limited by tangents through the origin: $T'(q)=T(q)/q$.
  - Little known in general for other a …or q! Possible: $\tau(q)>T(q)$
  - Proofs: Use Mass distortion Principle with factors M^q

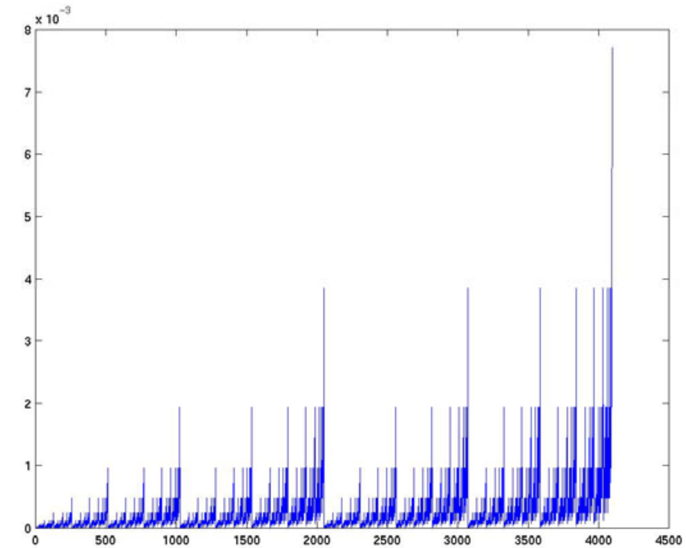
Wavelets for the Binomial

- Compactly supported wavelet
  - ΔI_n =wavelet coefficient corresponding to I_n
  - ΔI_n same rescaling property as measure itself
  - Same T(q)
  - Multifractal formalism holds

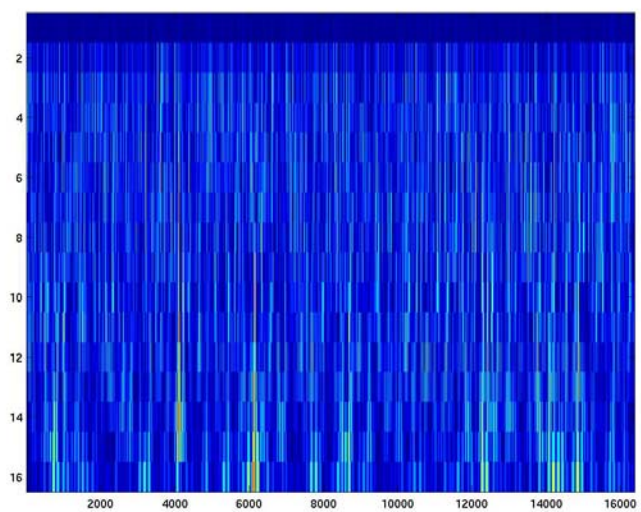
Toy examples

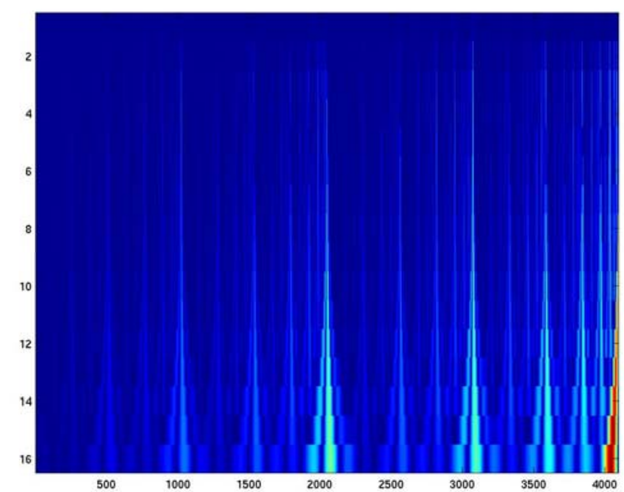


White noise



Cascade

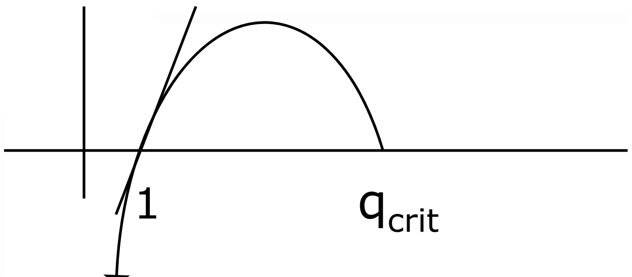




Log-Normal Binomial

- Deterministic envelope is a parabola: [Mandelbrot]

$$T(q) = (q-1)\left(1 - \frac{\sigma^2}{2\ln(2)} q\right) \quad \text{for } q < q_{\text{crit}} := 2\ln(2)/\sigma^2.$$



- Zeros: q=1, $q=q_{crit}$
- Non-Degeneracy: $T'(1) > 0 \Leftrightarrow q_{\text{crit}} > 1 \Leftrightarrow 2\ln(2) > \sigma^2$
- Spectrum is parabola as well

$$T^*(a) = 1 - \frac{\ln(2)}{2\sigma^2}\left(a - 1 - \frac{\sigma^2}{2\ln(2)}\right)^2$$

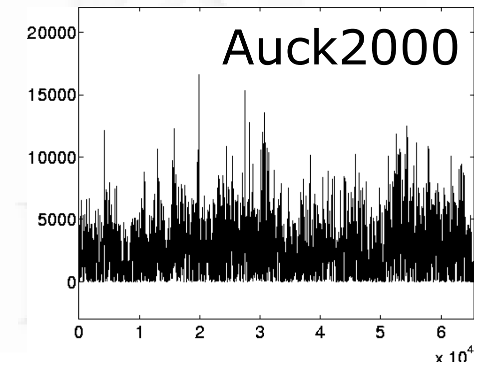
- Partition function τ(q) is non-decreasing,
- thus τ(q) > T(q) (at least) for $q>(1+q_{crit})/2$

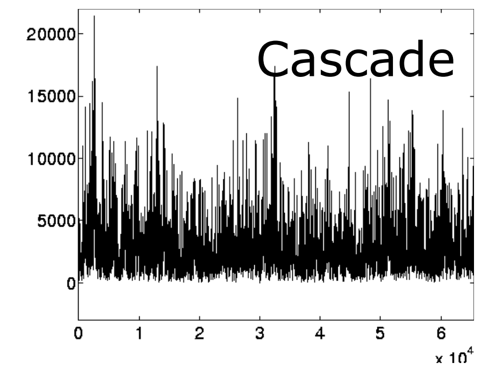
Multifractal Product of Pulses

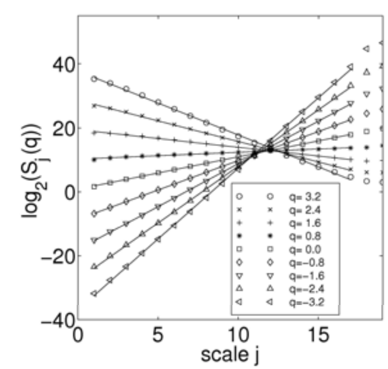
together with
I. Norros and P. Mannersalo

Network Traffic is Multifractal

- Visually striking
- Scaling of impressive quality
  (Levy Vehel & RR ’96,
  Norros & Mannersalo ’97,
  Willinger et al ‘98)
- Statistical models:
  – Binomial cascades with
    scale dependent multipliers
    (Crouse & RR ’98, Willinger et al ‘98)
- Not stationary!
  – Cumbersome for statistics
  – and probability (Queueing)







Multifractal paradigm

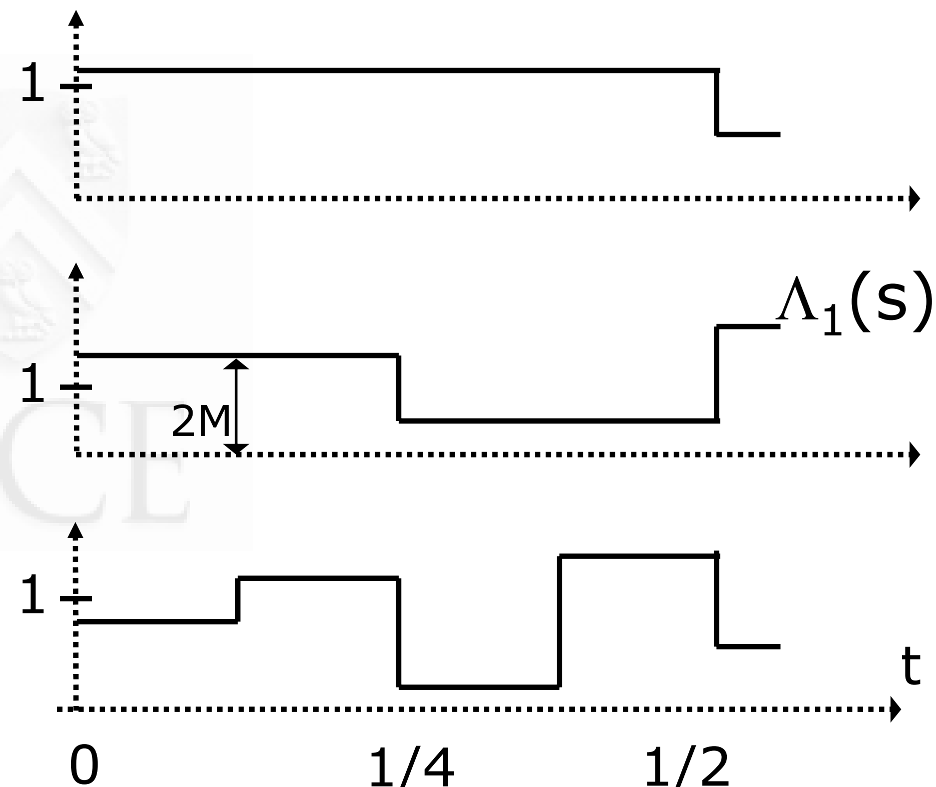
Multiplicative Processes:

- From redistributing mass to multiplying pulses

$$A(t) = \lim_{n\to\infty} \int_0^t \Lambda_0(s)\ldots\Lambda_n(s)ds$$

Binomial Cascade

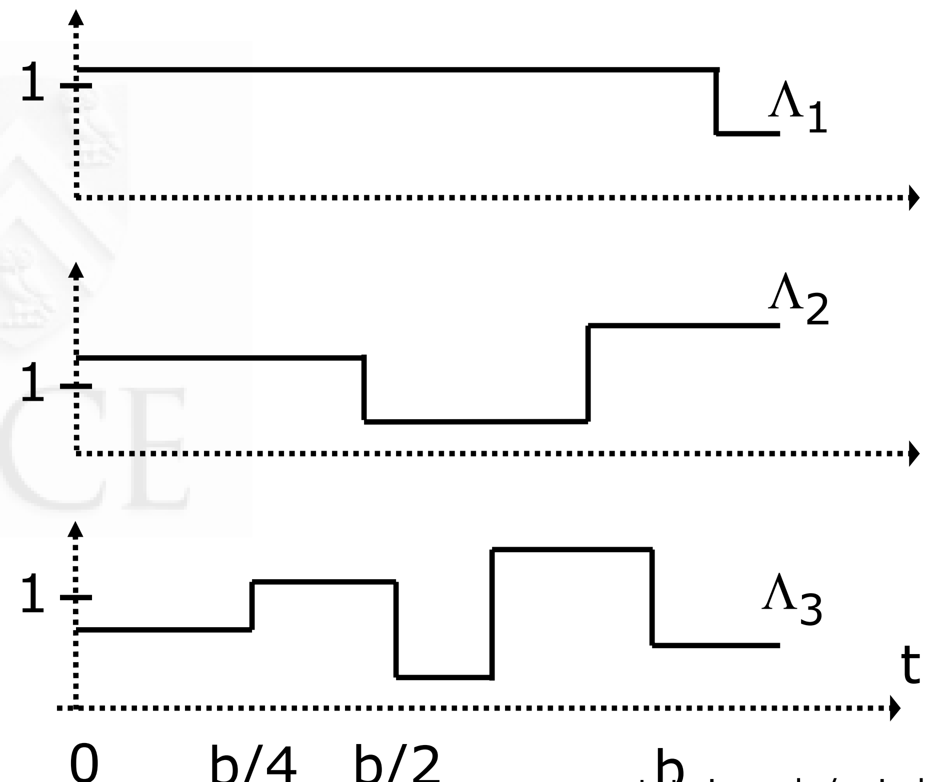
- $\Lambda_n(s)$ is constant on dyadic intervals
- Conservative:
$\Lambda_n(2k/2^n) + \Lambda_n((2k+1)/2^n) = 2$
- Martingale de Mandelbrot:
$E\,\Lambda_n(s) = 1$
- Not stationary



Multifractal paradigm

- Multiplicative Processes:

$$A(t) = \lim_{n\to\infty} \int_0^t \Lambda_0(s) \ldots \Lambda_n(s) ds$$



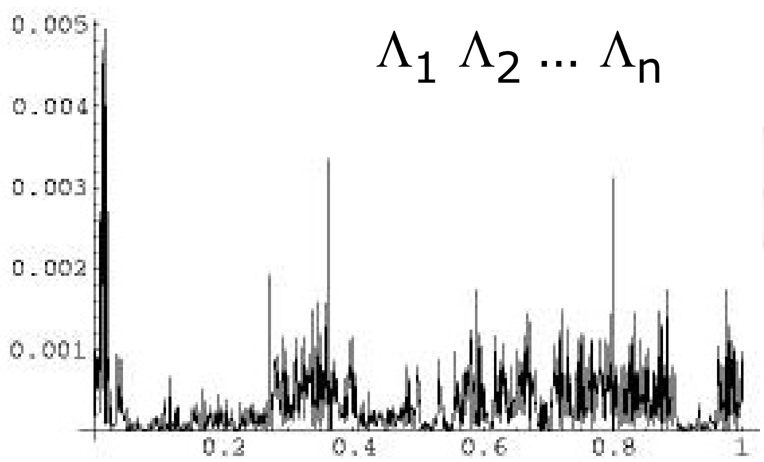
- Stationary Cascade
  - $\Lambda_n(s)$ is stationary
  - Conservation:

    $E\Lambda_n(t) = 1$
  - "self-similarity":

    $\Lambda_n(s) =_d \Lambda_1(sb^n)$

Parameters and Scaling

- Parameter estimation
  - $\Lambda_i(s)$: i.i.d. values with Poisson arrivals (λ_i):
  - $Z(s) = \log [\ \Lambda_1(s)\ \Lambda_2(s) \ldots \Lambda_n(s)\ ]$
  - $Cov(Z(t)Z(t+s)) = \Sigma_{i=1..n} \exp(-\lambda_i s) Var\ \Lambda_i(s)$
- Performance of predictors / simulations



$\Lambda_1\ \Lambda_2 \ldots \Lambda_n$

- Multifractal Envelope

(with Norros and Mannersalo)

$$T(q)=q-1-\log_2 E[\Lambda^q]$$

Interlude

Self-similar processes

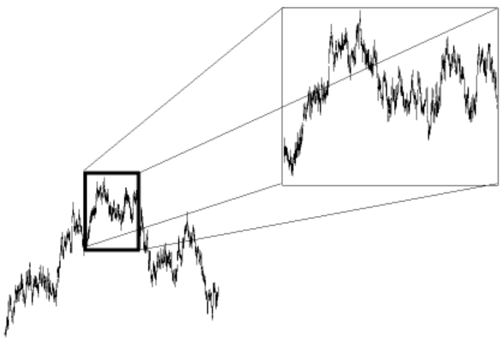
Statistical Self-similarity

- Self-similarity: canonical form
  - $B(at) =^{fdd} C(a)\,B(t)$ B: process, C: scale function
  - Iterate: $B(abt) =^{fdd} C(a)C(b)\,B(t)$
  - $C(a)C(b)=C(ab)$

  → $C(a) = a^H$: Powerlaw is default

- H-self-similar:

$$B(at) =^{fdd} a^H B(t)$$



- Examples
  - Gaussian: fractional Brownian motion $B_H(t)$ is unique H-self-similar Gaussian process with stationary increments.
  - Stable: not unique in general, a=1/H: Levy motion

Self-similar Processes

- What do they model?

  - Sustained excursions above/below the mean

- Different from (finite order) linear models
  - Auto-Regressive
  - ARMA
  - (G)ARCH
  - Exponential decay of correlations

- Corresponds to infinite order AR models
  - FARIMA
  - FIGARCH

$$fBm(t) = \int_{-\infty}^{t} K(t,s)\, dW(s)$$

Multifractal Subordination

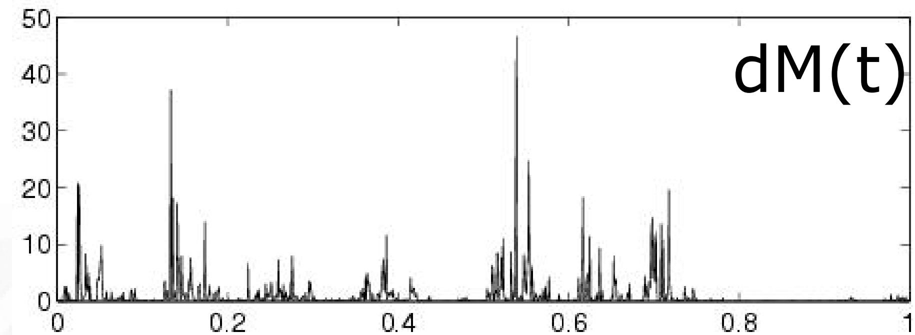
Processes with
multifractal oscillations

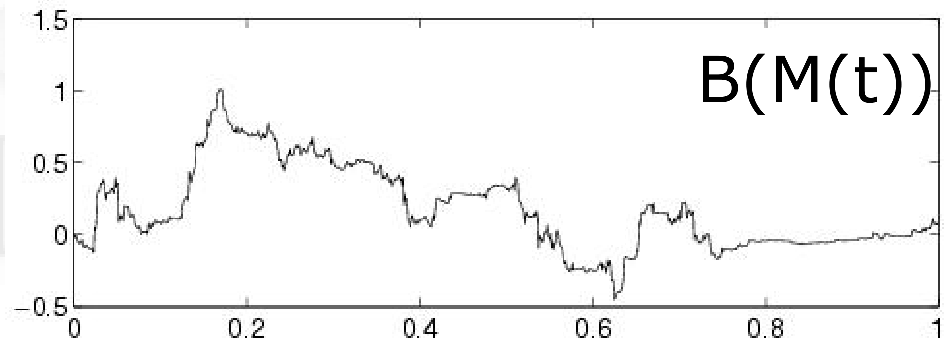
Multifractal time warp

$B_H(M(t))$: B_H fBm, dM independent measure

A versatile model

- $M(t)$: Multifractal
  Time change
  Trading time
- B: Brownian motion
  Gaussian fluctuations

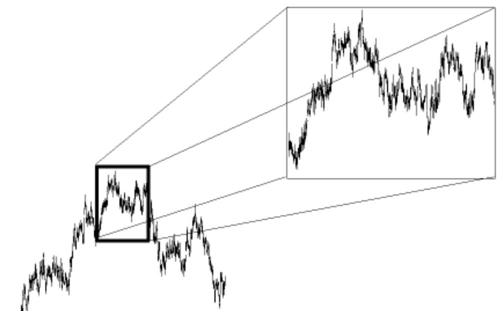




Hölder regularity

- Levy modulus of continuity:
  - With probability one for all t

$$|B_H(t+\delta) - B_H(t)| \simeq |\delta|^H$$

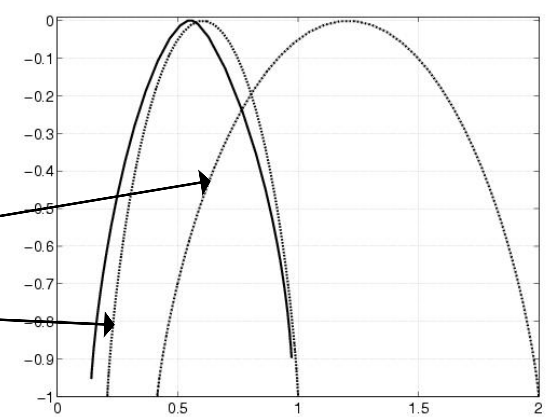


  - Thus, exponent gets stretched:

$$|B_H(M(t{+}\delta)){-}B_H(M(t))| \simeq |M(t{+}\delta){-}M(t)|^H \simeq |\delta|^{H\alpha(t)}$$

  - and spectrum gets squeezed:

$$\mathrm{dim} E_a[B_H(M)] = \mathrm{dim} E_{a/H}[M]$$



Multifractal formalism for $B_H(M(t))$

- Conditioning on M one finds:

$$\begin{aligned} \mathbb{E}|B_H(M(t+\delta)) - B_H(M(t))|^q &= \mathbb{E}|B_H(1)|^q \mathbb{E}|M(t+\delta) - M(t)|^{qH} \\ &\simeq |\delta|^{T_M(qH)} \end{aligned}$$

  – thus

$$T_{B(M)}(q) = T_M(qH)$$

  – which confirms the stretched exponent:

$$T'_{B(M)}(q) = H T'_M(qH)$$

  – and matches with warp formula before:

$$T^*_{B(M)}(a) = T^*_M(a/H)$$

  – If the formalism holds for M, then also for $B_H(M(t))$

Estimation: Wavelets decorrelate

(with P. Goncalves)

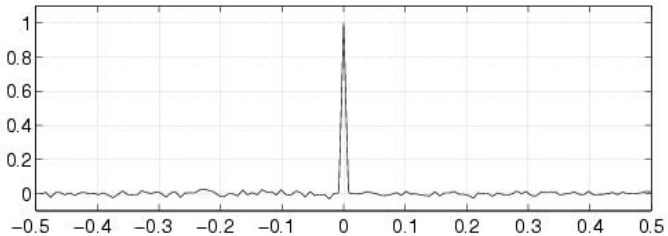
- $W_{jk} = \int \psi_{jk}(t)\, B(M(t))\, dt$
  N: number of vanishing moments

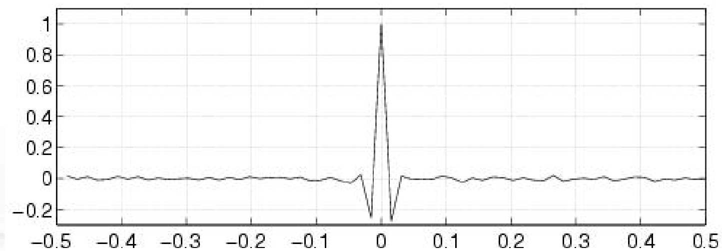
- $E[W_{jk}\, W_{jm}]$
  $= \int \int \Psi_{jk}(t)\, \Psi_{jm}(s)\, E[B(M(t))\, B(M(s))]\, dt\, ds$
  $= \int \int \Psi_{jk}(t)\, \Psi_{jm}(s)\, E[|M(t) - M(s)|^{2H}]\, dt\, ds$

  $\sim O(\, |k-m|^{T(2H) + 1 - 2N} \,) \quad (\, |k-m| \rightarrow \infty)$

Multifractal Estimation for B(M(t))

- Weak Correlations of Wavelet-Coefficients:
  (with P. Goncalves)





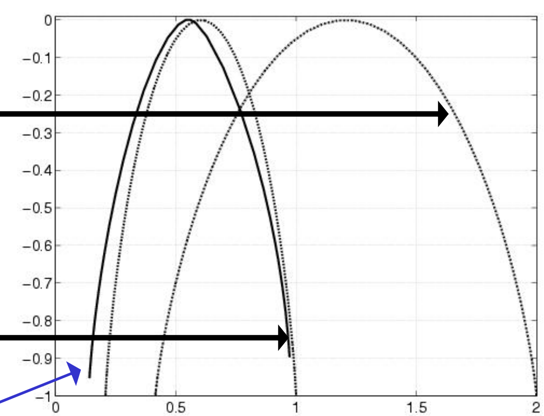
- Improved estimator due to weak correlations
- Multifractal Spectrum

$M(t+s) - M(t) \sim s^{a(t)}$

$B(t+u) - B(t) \sim u^{H} \quad (\forall\ t)$

$\rightarrow$

$B(M(t+s)) - B(M(t)) \sim s^{H*a(t)}$



From Multiplicative Cascades to Infinitely Divisible Cascades

with

P. Chainais and P. Abry

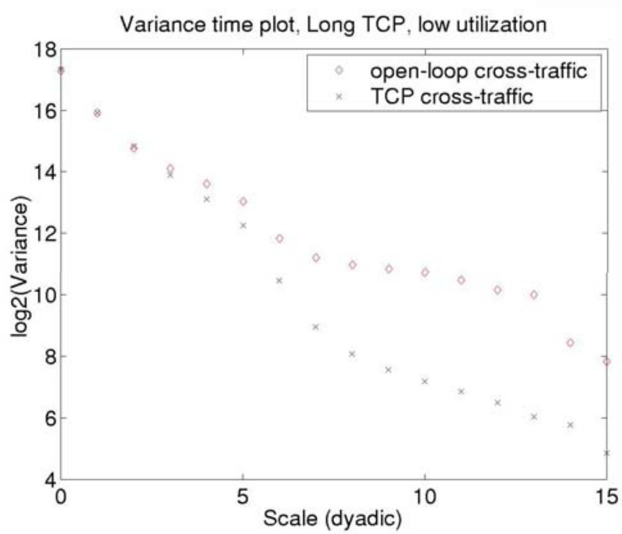
Independent work:
Castaing, Schmidt,
Barral-Mandelbrot, Bacry-Muzy

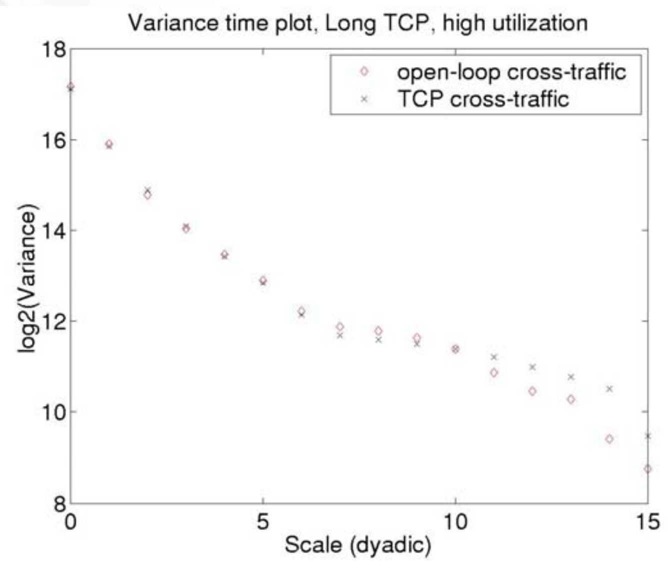
Adapting to the real world

Real world data

- can deviate from powerlaws: traffic
- has no preference for dyadic scales

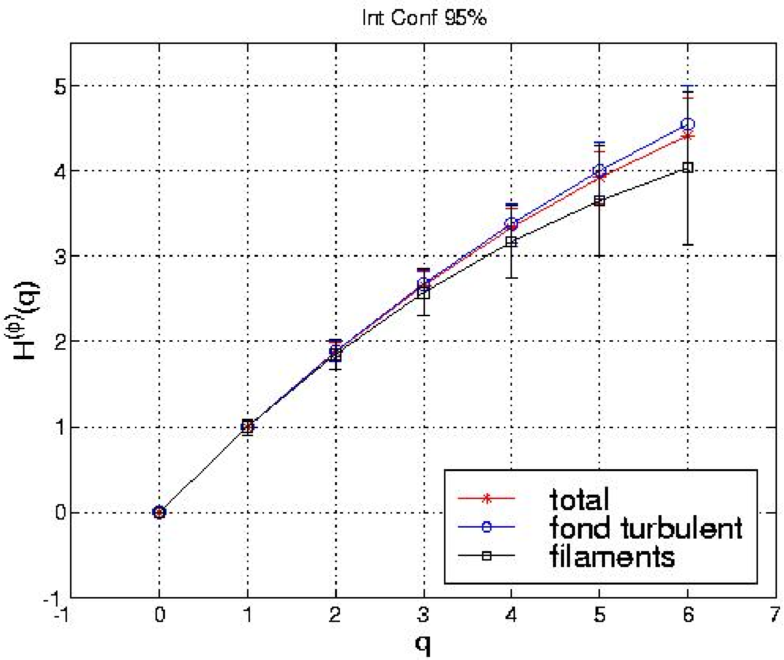
Lukacs: if the data does not fit to the model then too bad for the data.

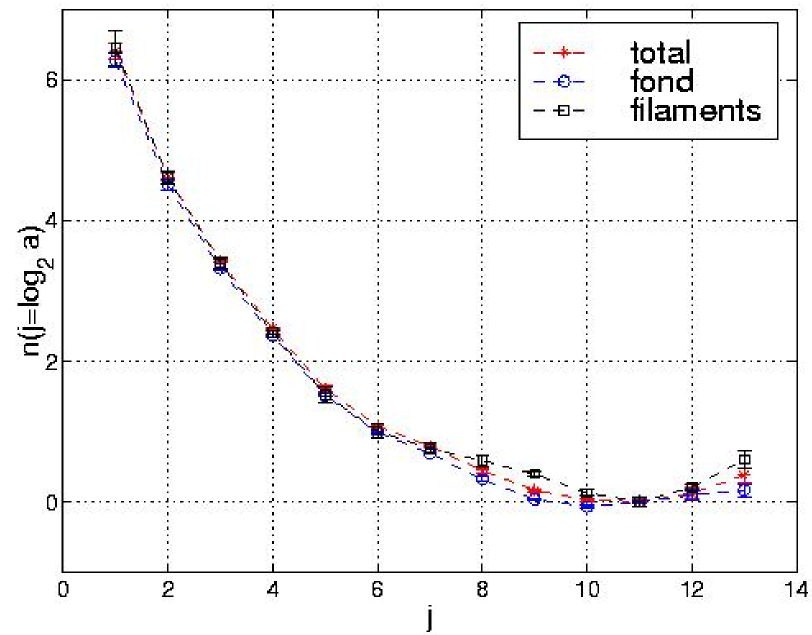




Experimental results

Courtesy P. Chainais





H(q)

n(a): non-powerlaw

Beyond Self-similarity

- Self-similarity revisited:
  - $B(at) =^d C(a)\, B(t)$ B: process, C: scale function
  - $B(abt) =^d C(a)C(b)\, B(t)$
  - $C(a)C(b)=C(ab) \rightarrow C(a) = a^H$
  - $E[|\, B(a^n)\, |^q] = c(q)\, (a^{qH})^n$
  - linear in q (mono-fractal)

Beyond Self-similarity

- Self-similarity revisited:
  - $B(at) =^d C(a)\, B(t)$ B: process, C: scale function
  - $B(abt) =^d C(a)C(b)\, B(t)$
  - $C(a)C(b)=C(ab) \;\rightarrow\; C(a) = a^H$
  - $E[|\, B(a^n)\, |^q] = c(q)\, (a^{qH})^n$
  - linear in q (mono-fractal)

- More flexible rescaling "Ansatz":
  - $C=C(a,t)$? : non-stationary increments
  - C=independent r.v. for every re-scaling :
  - $X(a\ldots at)= X(a^n t) = C_1(a)\ldots C_n(a)\, X(t)$: multiplicative
  - $E[|\, X(a^n)\, |^q] = c(q)\, E[|\, C(a)\, |^q]^n$
  - non-linear in q; powerlaw

Infinitely divisible scaling

$$
\begin{aligned}
\text{Self-similarity:} \quad & \mathbb{E}[|B(t+\delta) - B(t)|^q] &\simeq\ & \delta^{qH} \\
\text{Multifractal scaling:} \quad & \mathbb{E}[|M(t+\delta) - M(t)|^q] &\simeq\ & \delta^{1+T(q)} \\
\text{IDC scaling:} \quad & \mathbb{E}[|X(t+\delta) - X(t)|^q] &\simeq\ & \exp[n(\delta)\zeta(q)]
\end{aligned}
$$

- Multifractal scaling reduces to self-similarity if T is linear in q. (sometimes called mono-fractal)
- IDC reduces to multifractal scaling if n(δ)=-log(δ)
- In general n(δ) gives the speed of the cascade

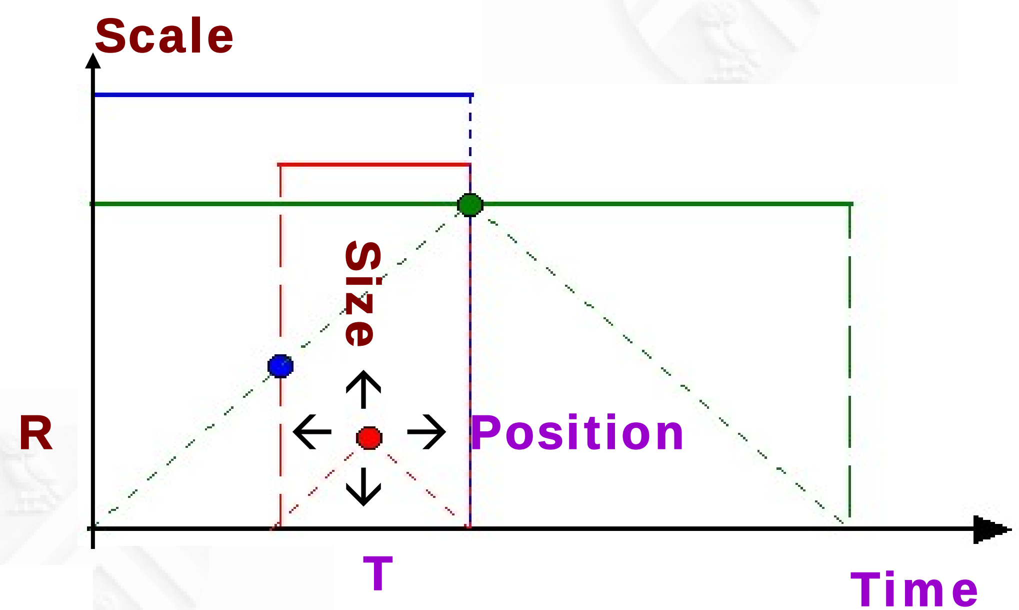
Geometry of Binomial Pulses

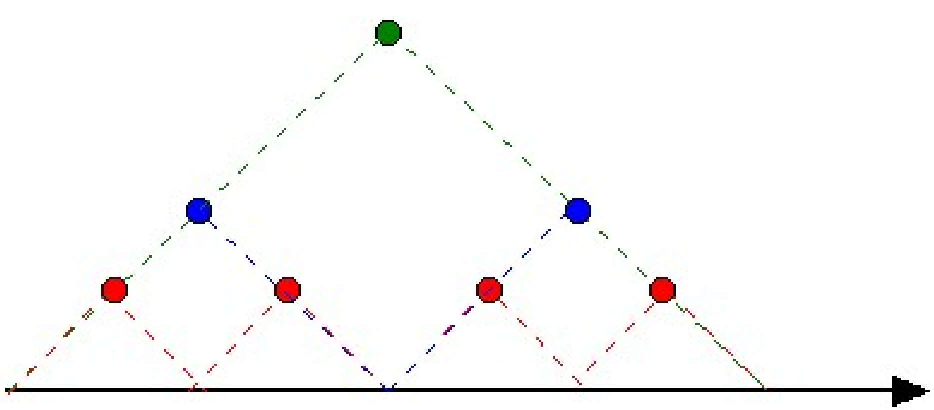
- Time-Scale plane: codes shape of pulses
  - **Position** (T=center)
  - **Size** (R=length)

Pulses:

$P_i(t) = W_i$ if $|t-t_i|<r_i/2$

$\qquad\quad 1$ else





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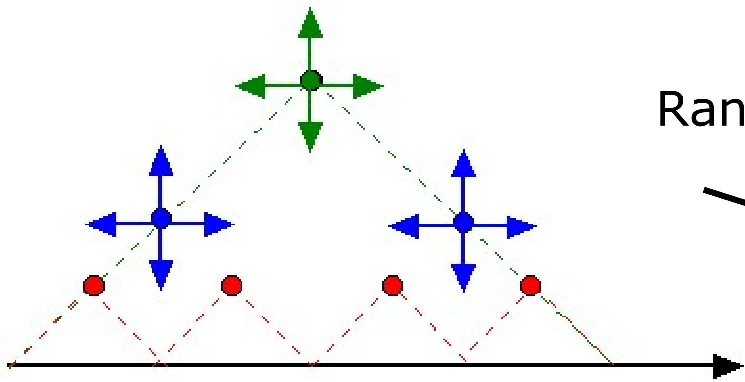
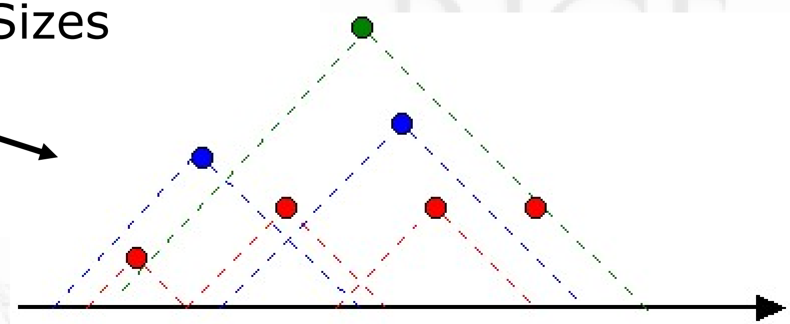
For Binomial: Strict dyadic geometry

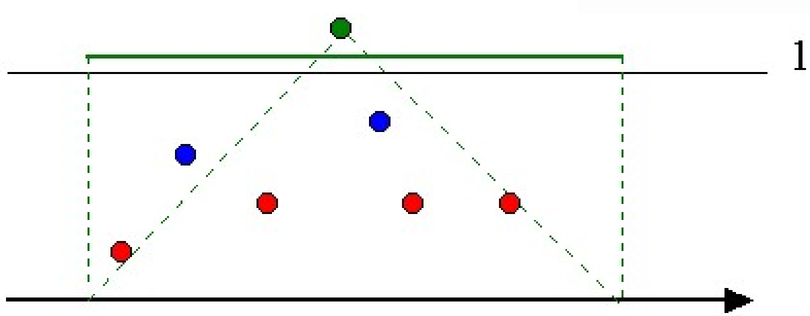
Stationary geometry



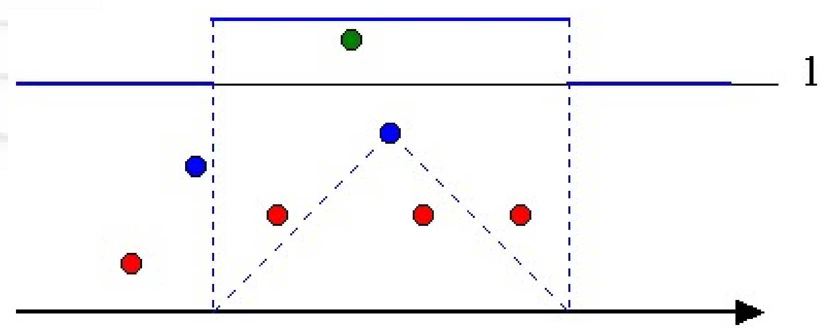
Randomize Positions
and Sizes



Large Scale Pulses



Medium Scale Pulses

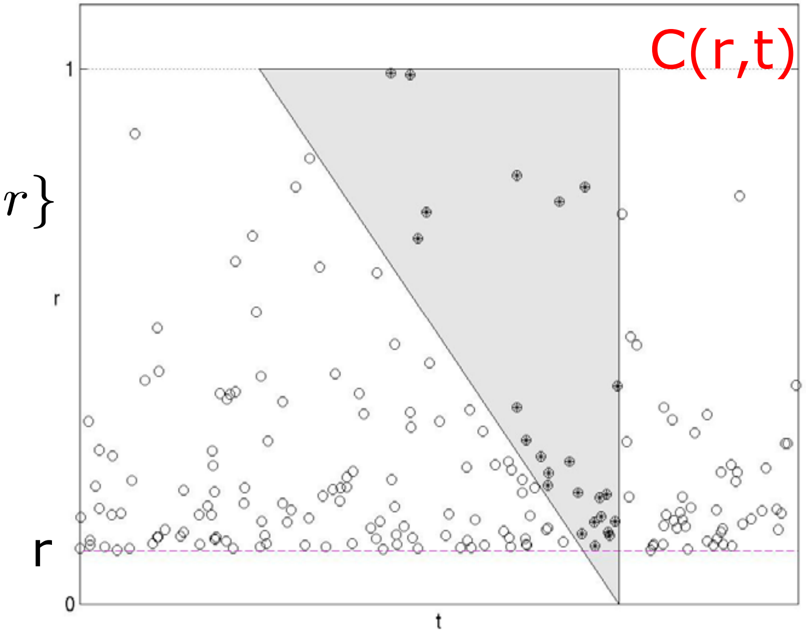


Compound Poisson Cascade

Poisson points (t_i , r_i) in time-scale plane with marks W_i

Cone of influence at t

$$C(r,t) = \{(t_i, r_i) : t - r_i < t_i < t, r_i > r\}$$



Cascade Process:

$$Q_r(t) = \prod_{(t_i,r_i)\in C(r,t)} W_i$$

- Poisson Cascades exhibit scaling properties akin to IDC scaling

$$m(\mathcal{C}(r,t)) = m(\mathcal{C}(r,0)) = \mathbb{E}[\#\{(t_i,r_i) \in \mathcal{C}(r,t)\}]$$

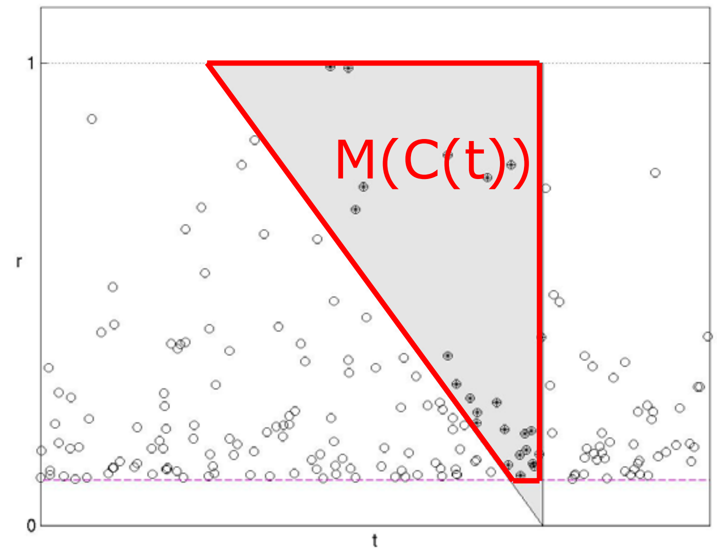
$$\mathbb{E}Q_r(t)^q = \exp\left[-\varphi(q) m(\mathcal{C}(r,*))\right]$$

Cascade and AR processes

- Continuous version (IDC):

$$\begin{aligned} Q(t) &= \exp M(C(t)) \\ &= \exp \int k_C(t,s)\,dM(s) \end{aligned}$$

  - M is an infinitely divisible measure



- Classic theory to be exploited:
  - AR-type processes

$$B_H(t) = \int \tilde{k}_H(s,t)\,dW(s)$$

  - kernel estimate of the random measure dM

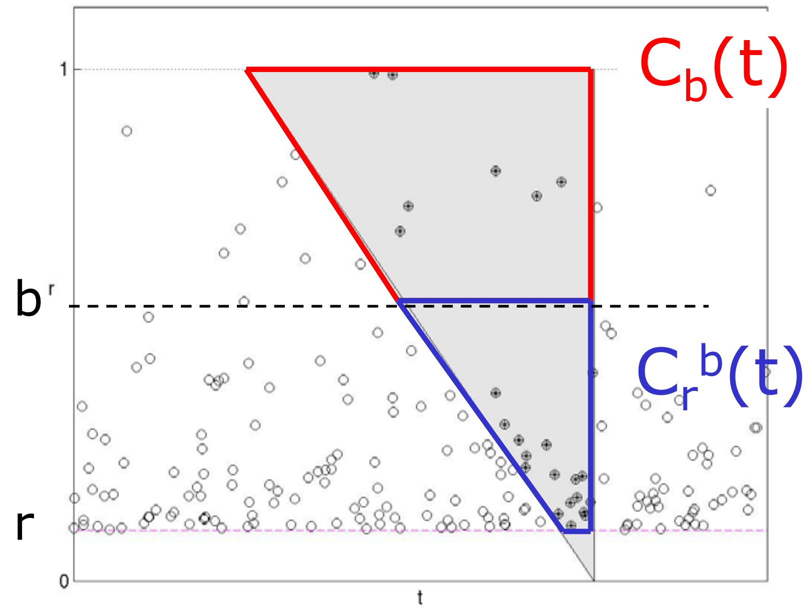
Cascades: Invariance and scaling

Infinitely divisible nature
and scaling of the cascade:

$$Q_r(t) = \prod_{C(r,t)} W_i = \underbrace{\prod_{C_b} W_i}_{} \times \underbrace{\prod_{C_r^b} W_i}_{}$$

$$= Q_b(t) \times \underbrace{\prod_{C_r^b} W_i}$$

Rescaled version of $Q_{r/b}$
in the scale-invariant case only!



Poisson Cascade has re-scaling properties;
in scale invariant case: akin to Product of Processes

Multifractal scaling

- Multifractal formalism holds in self-similar case [Barral-Mandelbrot]
- Infinitely Divisible Scaling

Recall

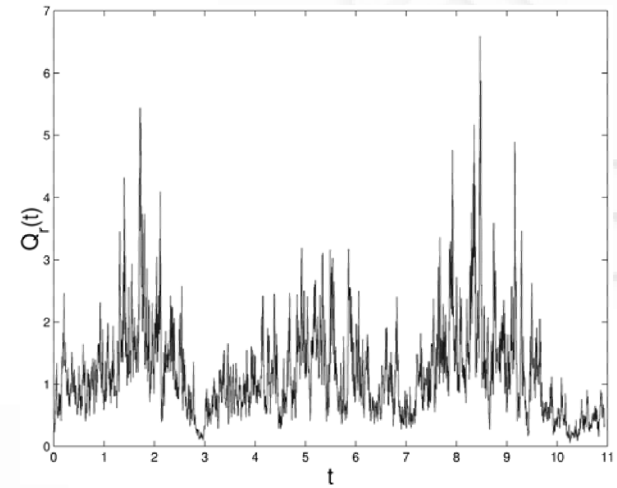
$$\mathbb{E} Q_r(t)^q = \exp\left[-\varphi(q) m(\mathcal{C}(r,*))\right]$$

$$\boxed{\mathbb{E} A(t)^q \simeq t^q \exp\left[-\varphi(q) m(\mathcal{C}(t,*))\right]}$$

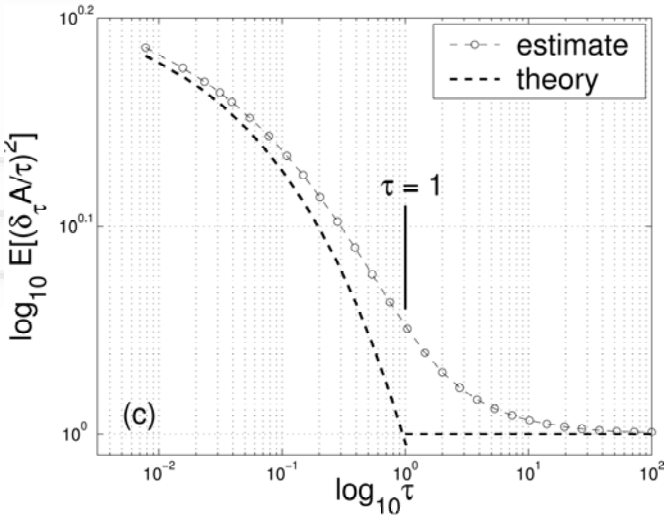
- powerlaw only if m(C(t,*)) = -log(t)
- for IDC in self-similar case [Bacry-Muzy,Barral]
- for CPC and log-normal IDC in certain non-powerlaw cases [Chainais-R-Abry]

Simulations

- Stationary Cascade:

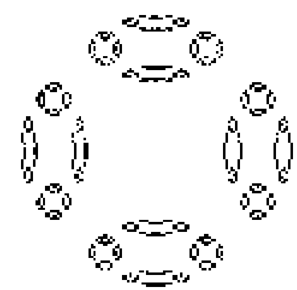


- Non-powerlaw scaling



“Never happy”: More flexibility

- Better control of scaling
- Wider range of known non-powerlaw scaling
- Higher dimensions: anisotropy
  - “As expected” in generic cases [Falconer, Olsen]
  - Formalism may break if directional preferences [McMullen, Bedford, Kingman, R]



Overall Lessons

- Multifractal spectrum <-> regularity
  - Besov spaces
  - Global Hoelder regularity
- Powerful modeling via multiplication through scales
  - Poisson product of Pulses
  - Multifractal warping
  - Degeneracy: price to pay for stationarity
- Estimation via wavelets
  - Multifractal envelopes
    - numerical $\tau(q)$,
    - Analytical $T(q)$
  - Choice of wavelet, of order q
  - Interpretation: what kind of spectrum did you estimate
    - Hoelder exponent
    - Wavelet decay

To take away

- Cascades matured to versatile multifractal models

- There remains much to do.

Reading on this talk

- www.stat.rice.edu/~riedi
- This talk
- Intro for the “untouched mind”
  - Explicit computations on Binomial
- Monograph on “Multifractal processes”
  - Multifractal formalism (proofs)
  - Multifractal subordination (warping)
- Papers, links