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Multifractal Processes

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ABSTRACT

This paper has two main objectives. First, it develops the multifractal formalism in a context suitable for both, measures and functions, deterministic as well as random, thereby emphasizing an intuitive approach. Second, it carefully discusses several examples, such as the binomial cascades and self-similar processes with a special eye on the use of wavelets. Particular attention is given to a novel class of multifractal processes which combine the attractive features of cascades and self-similar processes. Statistical properties of estimators as well as modelling issues are addressed.

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1 Introduction and Summary

Fractal processes have been instrumental in a variety of fields ranging from the theory of fully developed turbulence [73, 64, 36, 12, 7], to stock market modelling [28, 68, 69, 80], image processing [61, 21, 104], medical data [2, 98, 11] and geophysics [36, 65, 47, 92]. In networking, models using fractional Brownian motion (fBm) have helped advance the field through their ability to assess the impact of fractal features such as statistical self-similarity and long-range dependence (LRD) to performance [60, 81, 90, 89, 96, 34, 88].

Roughly speaking, a fractal entity is characterized by the inherent, ubiquitous occurrence of irregularities which governs its shape and complexity. The most prominent example is certainly fBm $B_H(t)$ [71]. Its paths are almost surely continuous but not differentiable. Indeed, the oscillation of fBm in any interval of size δ is of the order δ^H where $H \in (0,1)$ is the self-similarity parameter:

$$B_H(at) \stackrel{fd}{=} a^H B_H(t). \tag{1.1}$$

Reasons for the success of fBm as a model of LRD may be seen in the simplicity of its scaling properties which makes it amendable to analysis. The fact of being Gaussian bears further advantages. However, the scaling law (1.1) implies also that the oscillations of fBm at fine scales are uniform* which comes as a disadvantage in various situations (see Figure 1). Real world signals often possess an erratically changing oscillatory behavior (see Figure 2) which have earned them the name *multifractals*, but which also limits the appropriateness of fBm as a model. This rich structure at fine scales may serve as a valuable indicator, and ignoring it might mean to miss out on relevant information (see references above).

This paper has two objects. First, we present the framework for describing and detecting such a multifractal scaling structure. Doing so we survey local and global multifractal analysis and relate them via the multifractal formalism in a stochastic setting. Thereby, the importance of higher order statistics will become evident. It might be especially appealing to the reader to see wavelets put to efficient use. We focus mainly on the analytical computation of the so-called multifractal spectra and on their mutual relations. Thereby, we emphasize issues of observability by striving for formulae which hold for all or almost all paths and by pointing out the necessity of oversampling needed to capture certain rare events. Statistical properties of estimators of multifractal quantities as well as modelling issues are addressed elsewhere (see [41, 3, 40] and [68, 89, 88]).

Second, we carefully discuss basic examples as well as *Brownian motion in multifractal time*, $B_{1/2}(\mathcal{M}(t))$. This process has recently been suggested as a model for stock market exchange by Mandelbrot who argues that oscillations in exchange rates occur in multifractal 'trading time' [68, 69]. With the theory developed in this paper, it becomes an easy task to explore $B_{1/2}(\mathcal{M}(t))$ from the multifractal point of view, and with

*This property is also known as the Lv́ey modulus of continuity in the case of Brownian motion. For fBm see [5, Thm. 8.3.1.].

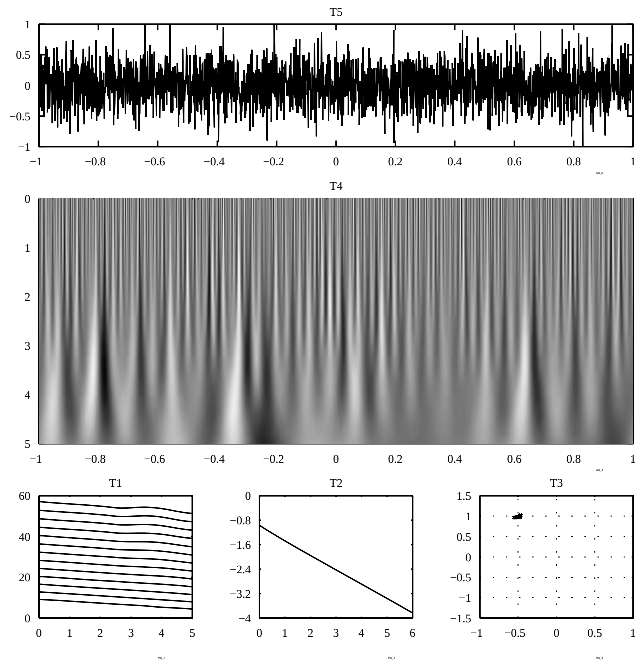

FIGURE 1. *Fractional Brownian motion, as well as its increment process called fGn (displayed on top in T5), has only one singularity exponent* $h(t) = H$*, a fact which is represented in the linear partition function* τ *(see T2) and a multifractal spectrum (see T3) which consists of only one point: for fBm* $(H, 1)$ *and for fGn* $(H - 1, 1)$*. For further details on the plots see (1.9), (1.6) and Figure 7.*

little more effort also more general multifractal 'subordinators'. The reader interested in these multifractal processes may wish, at least at first reading, to content himself with the notation introduced on the following few pages, skip the sections which deal more carefully with the tools of multifractal analysis, and proceed directly to the last sections. The remainder of this introduction provides a summary of the contents of the paper, following roughly its structure.

1.1 Singularity Exponents

In this work, we are mainly interested in the geometry or local scaling properties of the paths of a process $Y(t)$. Therefore, all notions and results concerning paths will apply to functions as well. The study of fine scale properties of functions (as opposed to measures) has been pioneered in the work of Arneodo, Bacry and Muzy [7, 78, 79, 1, 2, 80], who were also the first to introduce wavelet techniques in this context. For simplicity of the presentation we take $t \in [0, 1]$. Extensions to the real line $\mathbb{R}$ as well as to higher dimensions, being straightforward in most cases, are indicated.

A typical feature of a fractal process $Y(t)$ is that it has a non-integer degree of differentiability, giving rise to an interesting analysis of its local *Hölder exponent* $H(t)$

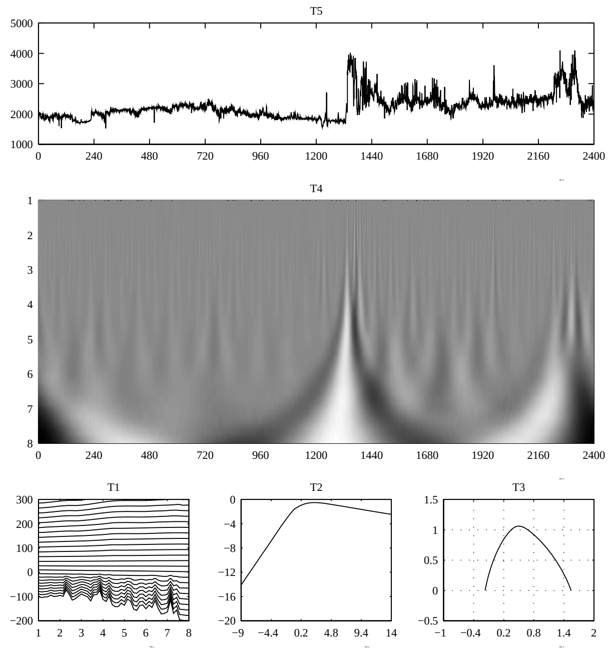

FIGURE 2. *Real world signals such as this geophysical data often exhibit erratic behavior and their appearance may make stationarity questionable. One such feature are 'trends' which sometimes can be explained by strong correlations (LRD). Another such feature are the sudden jumps or 'bursts' which in turn are a typical for multifractals. For such signals the singularity exponent $h(t)$ depends erratically on time t, a fact which is reflected in the concave partition function τ (see T2) and a multifractal spectrum (see T3) which extends over a non-trivial range of singularity exponents.*

which is roughly defined through

$$|Y(t') - P(t')| \simeq |t' - t|^{H(t)} \tag{1.2}$$

for some polynomial P which in nice cases is simply the Taylor polynomial of Y at t. A rigorous definition is given in (2.1).

Provided the polynomial is *constant*, $H(t)$ can be obtained from the limiting behavior of the so-called *coarse Hölder exponents*, i.e.,

$$h_\varepsilon(t) = \frac{1}{\log \varepsilon} \log \sup_{|t'-t|<\varepsilon} |Y(t') - Y(t)|. \tag{1.3}$$

For rigorous statements we refer to (2.2) and lemma 2.3.

However, as the example $t^2 + t^{2.7}$ shows, the use of $h_\varepsilon(t)$ is ineffective in the presence of polynomial trends. Then, $h_\varepsilon(t)$ will reflect the lowest non-constant term of the Taylor polynomial of Y at t. For this reason, and also to avoid complications introduced through the computation of the supremum in (1.3), one may choose to employ *wavelet* *decompositions* or other tools of time frequency analysis. Properly chosen wavelets are blind to polynomials and due to their scaling properties they contain information on the Hölder regularity of Y [51, 23]. Their application in multifractal estimation has been pioneered by [7, 53, 30]. Furthermore, wavelets provide unconditional basis for several regularity spaces such as Besov spaces (see (2.14) and (6.2)) whence their use bears further advantages.

Yet, the 'classical' choice of a singularity exponent is

$$\alpha_k^{(n)} = \frac{1}{-n \log 2} \log \big(\mathcal{M}((k+1)2^{-n}) - \mathcal{M}(k2^{-n})\big). \tag{1.4}$$

It is attractive due to its simplicity and becomes actually quite powerful when studying monotonously increasing processes $\mathcal{M}(t)$, in particular the distribution functions of *singular measures*, such as cascades.

In this chapter we will introduce the exponents $w_k^{(n)}$ emerging from a wavelet based analysis and elaborate on the relation between these different singularity exponents $h_k^{(n)}$, $\alpha_k^{(n)}$ and $w_k^{(n)}$.

1.2 Multifractal Spectra

As indicated we are mainly interested in the geometry or local regularity of the paths of $Y(t)$. Let us fix such a realization for the time-being.

Local analysis

Ideally, one would like to quantify in geometrical as well as statistical sense which values $H(t)$ appear on a given path of the process Y, and how often one will encounter them. Towards the first description one studies the sets

$$E_h^{[a]} = \{t : H(t) = a\} \tag{1.5}$$

for varying a. Similarly, one could consider sets $E_\alpha^{[a]}$ and $E_w^{[a]}$ defined through the limiting behavior of the singularity exponents $\alpha_k^{(n)}$ or $w_k^{(n)}$, respectively. If no confusion regarding the choice of $h_k^{(n)}$, $w_k^{(n)}$ or $\alpha_k^{(n)}$ can arise, we simply drop the index.

The sets $E^{[a]}$ form a decomposition of the support of Y according to its singularity exponents. We say that Y has a *rich multifractal structure* if these sets $E^{[a]}$ are highly interwoven, each lying dense on the line. Typically, only one of the $E^{[a]}$ has full Lebesgue measure, while the others form dusts, more precisely, sets with non-integer Hausdorff dimension $\dim(E^{[a]})$ [32]. Dimensions are always positive, and the smaller the dimension of a set the 'thinner' the set. In this sense, the function

$$a \mapsto \dim(E^{[a]}) \tag{1.6}$$

gives a compact representation of the complex singularity structure of Y. It has been termed the *multifractal spectrum* of Y and is studied extensively in the 'classical' literature.

To develop some intuition let us consider a differentiable path. To avoid trivialities let us assume that this path and its derivative have no zeros. Then, $\dim(E^{[a]})$-spectrum reduces to the point $(1,1)$. On the other hand, if $H(t)$ is continuous and not constant on intervals then each $E^{[a]}$ is finite and $\dim(E^{[a]})=0$ for all a in the range of $H(t)$. A spectrum $\dim(E^{[a]})$ with non-degenerate form is, thus, indeed indication for rich singularity behavior. By this we mean that $H(t)$ changes erratically with t and takes each value a on a rather large set $E^{[a]}$.

Global analysis

A simpler way of capturing the complex structure of a signal is obtained when adapting the concept of box-dimension to the multifractal context. As the name indicates, one aims at an estimate of $\dim(E^{[a]})$ by counting the intervals – or boxes – over which Y increases roughly with the 'right' Hölder exponent. Therefore, we need to introduce grain exponents, a discrete approximation to $h_\varepsilon(t)$ (see (1.3)):

$$h_k^{(n)} := -(1/n)\log_2 \sup\{|Y(s)-Y(t)| \,:\, (k-1)2^{-n} \le s \le t \le (k+2)2^{-n}\} \tag{1.7}$$

and define the *grain (multifractal) spectrum* as [73, 46, 45, 91]

$$f(a) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{\log N^{(n)}(a,\varepsilon)}{n\log 2}, \tag{1.8}$$

where $N^{(n)}(a,\varepsilon) = \#\{k \,:\, |h_k^{(n)}-a|<\varepsilon\}$ counts, how many of the grain exponents $h_k^{(n)}$ are approximately equal to a. Similarly, one may define such spectra for the singularity exponents $\alpha_k^{(n)}$ and $w_k^{(n)}$. If confusion may arise, we will indicate the chosen exponent by writing explicitly $f_h(a)$, $f_\alpha(a)$, or $f_w(a)$.

This multifractal spectrum can be interpreted (at least) in three ways. First, as mentioned already it is related to the notion of *dimensions*. Indeed, a simple argument shows that $\dim(E^{[a]}) \le f(a)$ [94]. The essential ingredient for a proof is the fact that the calculation of $\dim(E^{[a]})$ involves finding an optimal covering of $E^{[a]}$ while $f(a)$ considers only uniform covers. In short, $f(a)$ provides an upper bound on the dimension and, thus, the 'size' of the sets $E^{[a]}$.

Second, (1.8) suggests that the *re-normalized histograms* $(1/n)\log_2 N^{(n)}(a,\varepsilon)$ should all be roughly equal at small scales 2^{-n} to the *scale independent* $f(a)$. It should be remembered that this is foremost (by definition) a property of the *paths* of the given process. We stress this point because it is tempting to argue that –at least under suitable ergodicity assumptions– one should see the marginal distribution of $h_k^{(n)}$ reflected in f. However, one should not overlook that the logarithmic re-normalization implemented in $f(a)$ is aimed at detecting exponential scaling properties rather than the marginals on multiple scales themselves. For fBm (see (1.1)) this re-normalization indeed causes all details of the Normal multi-scale marginals to be washed out into a virtually structureless $f(a)$ which gives notice of the presence of only *one* scaling law, the self-similarity (1.1) with parameter H. Thus, f expresses that fBm is 'mono-fractal', as mentioned above. To the contrary with 'multi-fractal' processes such as multiplicative cascades, for which f reflects the presence of an entire range of scaling exponents (see (5.32)).

The third natural context for the coarse spectrum f is that of Large Deviation Principles (LDP) [29, 91]. Indeed, $N^{(n)}(a,\varepsilon)/2^n$ defines a probability distribution† on $\{h_k^{(n)} : k = 0,\ldots,2^n-1\}$. Alluding to the Law of Large Numbers (LLN) we may expect this distribution to be concentrated more and more around the 'most typical' or 'expected' value as n increases. The spectrum $f(a)$ measures how fast the chance $N^{(n)}(a,\varepsilon)/2^n$ to observe a 'deviant' value a decreases, i.e., $N^{(n)}(a,\varepsilon)/2^n \simeq 2^{f(a)-1}$.

The close connection to LDP leads one to study the scaling of 'sample moments' through the so-called *partition function* [45, 46, 36, 91]

$$\tau_h(q) := \liminf_{n\to\infty} \frac{\log S_h^{(n)}(q)}{-n\log 2} \qquad \text{where} \quad S_h^{(n)}(q) := \sum_{k=0}^{2^n-1} 2^{-nqh_k^{(n)}}, \tag{1.9}$$

which are defined for all $q \in \mathbb{R}$. Similarly, replacing $h_k^{(n)}$ by $\alpha_k^{(n)}$, one defines $\tau_\alpha(q)$ and $S_\alpha^{(n)}(q)$. The latter takes on the well-known form of a partition sum

$$S_\alpha^{(n)}(q) = 2^{-nq\alpha_k^{(n)}} = \sum_{k=0}^{2^n-1} \left|Y\left((k+1)2^{-n}\right) - Y(k2^{-n})\right|^q. \tag{1.10}$$

Again similarly, one defines $\tau_w(q)$ and $S_w^{(n)}(q)$ by replacing $h_k^{(n)}$ by wavelet based exponents $w_k^{(n)}$ (see (2.11)). Again, if no confusion on the choice of $h_k^{(n)}$, $w_k^{(n)}$ or $\alpha_k^{(n)}$ can arise, we simply drop the index h, α or w .

1.3 Multifractal Formalism

The theory of LDP suggests $f(a)$ and $\tau(q)$ are strongly related since $2^{-n}S^{(n)}(q)$ is the moment generating function of the random variable $A_n(k) := -nh_k^{(n)}\ln(2)$ (recall footnote †). For a motivation of a formula connection $f(a)$ and $\tau(q)$ consider the heuristics

$$S^{(n)}(q) \;=\; \sum_a \sum_{h_k^{(n)}\simeq a} 2^{-nqh_k^{(n)}} \simeq \sum_a 2^{nf(a)}2^{-nqa} = \sum_a 2^{-n(qa-f(a))} \simeq 2^{-n\inf_a(qa-f(a))}.$$

Assuming that $\sum_a$ has only finite many terms the last step simply replaces the sum by its strongest term. Making this entire argument rigorous we prove in this paper that

$$\tau(q) = f^*(a) := \inf_a (qa - f(a)). \tag{1.11}$$

Here $(\cdot)^*$ denotes the Legendre transform which is omnipresent in the theory of LDP. Indeed, by applying a theorem due to Gärtner and Ellis [27] and imposing some regularity on $\tau(q)$ theorem 3.5 shows that the family of probability densities defined by $N^{(n)}(a,\varepsilon)/2^n$ satisfies the *full LDP [26] with rate function* f meaning that f is actually a double-limit and $f(a) = \tau^*(a)$. Corollary 4.5 establishes that always

$$f(a) = \tau^*(a) = qa - \tau(q) \qquad \text{at points } a = \tau'(q). \tag{1.12}$$

†Recall that we fixed a path of Y. Randomness is here understood in choosing k.

Going through some of the explicitly calculated examples in Section 5.5 will help demystify the Legendre transform. A tutorial on the Legendre transform in contained in Appendix A of [89].

From (1.11) follows, that $f(a) \le f^{**}(a) = \tau^*(a)$ and also that $\tau(q)$ is a concave function, hence continuous and almost everywhere differentiable.

1.4 Deterministic Envelope

So far, all that has been said applies to any given function or path of a process. In the random case, one would often like to use a simple analytical approach in order to gain intuition or an estimate of f for a typical path of Y.

To this end we formulate a LDP for the sequence of distributions of $\{h_k^{(n)}\}$ where randomness enters now through choosing $k \in \{0, \dots, 2^n - 1\}$ as well as through the randomness of the process itself, i.e., through $Y_t(\omega)$ where ω lies in the probability space (Ω, P_Ω). The moment generating function of $A_n(k, \omega) = -nh_k^{(n)}(\omega)\ln(2)$ with k and ω random is $2^{-n}\mathbb{E}_\Omega[S^{(n)}(q)]$. This leads to defining the 'deterministic envelope':

$$T(q) := \liminf_{n\to\infty} \frac{-1}{n} \log_2 \mathbb{E}_\Omega S^{(n)}(q) \tag{1.13}$$

and the corresponding 'rate function' F (see (3.23)). As with the pathwise $f(a)$ and $\tau(q)$ we have here again $T(q) = F^*(q)$. More importantly, it is easy to show that $\tau(q, \omega) \ge T(q)$ almost surely (see lemma 3.9). Thus:

Corollary 1.1. *With probability one the multifractal spectra are ordered as follows: for all a*

$$\dim(E^{[a]}) \le f(a) \le \tau^*(a) \le T^*(a), \tag{1.14}$$

provided that they are all defined in terms of the same singularity exponent.

Great effort has been spent on investigating under which assumptions equality holds between some of the spectra, as a matter of fact mostly between spectra based on different scaling exponents. Indeed, the most interesting combinations seem to be $\dim(E^{[a]})$ with scaling exponents $h_k^{(n)}$ and $\alpha_k^{(n)}$, and $\tau^*(a)$ with scaling exponents $w_k^{(n)}$ and $\alpha_k^{(n)}$, the former for its importance in the analysis of regularity, the latter for its numerical relevance. It has become the accepted term in the literature to say that *the multifractal formalism holds* if any such spectra are equal; indeed they are in a generic sense [52]. However, this terminology might sometimes be confusing if the nature of the parts of such an equality is not indicated. We prefer here to call (1.14) *the multifractal formalism*: this formula 'holds' for any fixed choice of a singularity exponent as is shown in the paper.

1.5 Self-similarity and LRD

The statistical self-similarity as expressed in (1.1) makes fBm, or rather its increment process a paradigm of *long range dependence* (LRD). To be more explicit let δ denote some fixed lag and define *fractional Gaussian noise* (fGn) as

$$G(k) := B_H((k+1)\delta) - B_H(k\delta). \tag{1.15}$$

Having the LRD property means that the auto-correlation $r_G(k) := \mathbb{E}_\Omega[G(n+k)G(n)]$ decays so slowly that $\sum_k r_G(k) = \infty$. The presence of such strong dependence bears an important consequence on the aggregated processes

$$G^{(m)}(k) := \frac{1}{m} \sum_{i=km}^{(k+1)m-1} G(i). \tag{1.16}$$

They have a much higher variance, and variability, than would be the case for a short range dependent process. Indeed, if $X(k)$ are i.i.d., then $X^{(m)}(k)$ has variance $(1/m^2)\mathrm{var}(X_0+\ldots+X_{m-1}) = (1/m)\mathrm{var}(X)$. For G we find, due to (1.1) and $B_H(0) = 0$,

$$\mathrm{var}(G^{(m)}(0)) = \mathrm{var}\left(\frac{1}{m}B_H(m\delta)\right) = \mathrm{var}\left(\frac{m^H}{m}B_H(\delta)\right) = m^{2H-2}\mathrm{var}\left(G(0)\right). \tag{1.17}$$

For $H > 1/2$ this expression decays indeed much slower than $1/m$. As is shown in [19] $\mathrm{var}(X^{(m)}) \simeq m^{2H-2}$ is equivalent to $r_X(k) \simeq k^{2H-2}$ and so, $G(k)$ is indeed LRD for $H > 1/2$ (this follows also directly from (7.3)).

Let us demonstrate with fGn how to relate LRD with multifractal analysis using only that it is a zero-mean processes, not (1.1). To this end let $\delta = 2^{-n}$ denote the finest resolution we will consider, and let 1 be the largest. For $m = 2^i$ $(0 \le i \le n)$ the process $mG^{(m)}(k)$ becomes simply $B_H((k+1)m\delta) - B_H(km\delta) = B_H((k+1)2^{i-n}) - B_H(k2^{i-n})$. But the second moment of this expression —which is also the variance— is exactly what determines $T_\alpha(2)$ (compare (1.10)). More precisely, using stationarity of G and substituting $m = 2^i$, we get

$$2^{-(n-i)T_\alpha(2)} \simeq \mathbb{E}_\Omega\left[S_\alpha^{n-i}(2)\right] = \sum_{k=0}^{2^{n-i}-1} \mathbb{E}_\Omega\left[|mG^{(m)}(k)|^2\right] = 2^{n-i}2^{2i}\mathrm{var}\left(G^{(2^i)}\right). \tag{1.18}$$

This should be compared with the definition of the LRD-parameter H via

$$\mathrm{var}(G^{(m)}) \simeq m^{2H-2} \qquad \text{or} \qquad \mathrm{var}(G^{(2^i)}) = 2^{i(2H-2)}. \tag{1.19}$$

At this point a conceptual difficulty arises. Multifractal analysis is formulated in the limit of small scales ($i \to -\infty$) while LRD is a property at large scales ($i \to \infty$). Thus, the two exponents H and $T_\alpha(2)$ can in theory only be related when assuming that the scaling they represent is actually exact at all scales, and not only asymptotically. When this assumption is violated, the two approaches may provide strikingly different answers (compare Example 7.2).

In any real world application, however, one will determine both, H and $T_\alpha(2)$, by finding a *scaling region* $\underline{i} \le i \le \bar{i}$ in which (1.18) and (1.19) hold up to satisfactory precision. Comparing the two scaling laws in i yields $T_\alpha(2) + 1 - 2 = 2H - 2$, or

$$H = \frac{T_\alpha(2)+1}{2}. \tag{1.20}$$

This formula expresses most pointedly, how *multifractal analysis goes beyond second order statistics*: in (1.26) we compute with $T(q)$ the scaling of *all* moments. The formula (1.20) is derived here for zero-mean processes, but can be put on more solid grounds using wavelet estimators of the LRD parameter [4] which are more robust than the ones obtained through variance of the increment process. The same formula (1.20) reappears also for multifractals, suggesting that it has some 'universal truth' to it, at least in the presence of 'perfect scaling' (see (1.29) and (7.25), but also Example 7.2).

1.6 Multifractal Processes

The most prominent examples where one finds coinciding, strictly concave multifractal spectra are the distribution functions of *cascade* measures [64, 56, 15, 33, 6, 82, 49, 91, 95, 86] for which $\dim(E^{[a]})$ and $T^*(a)$ are equal and have the form of a $\cap$ (see Figure 6 and also 3 (e)). These cascades are constructed through some multiplicative iteration scheme such as the binomial cascade, which is presented in detail in the paper with special emphasis on its wavelet decomposition. Having positive increments, however, this class of processes is sometimes too restrictive. fBm, as noted, has the disadvantage of a poor multifractal structure and does not contribute to a larger pool of stochastic processes with multifractal characteristics.

It is also notable that the first 'natural', truly multifractal stochastic process to be identified was Lévy motion [54]. This example is particularly appealing since scaling is not injected into the model by an iterative construction (this is what we mean by the term natural). However, its spectrum is, though it shows a non-trivial range of singularity exponents $H(t)$, degenerated in the sense that it is linear.

Construction and Simulation

With the formalism presented here, the stage is set for constructing and studying new classes of truly multi-fractional processes. The idea, to speak in Mandelbrot's own words, is inevitable after the fact. The ingredients are simple: a multifractal 'time warp', i.e., an increasing function or process $\mathcal{M}(t)$ for which the multifractal formalism is known to hold, and a function or process V with strong mono-fractal scaling properties such as *fractional Brownian motion* (fBm), a Weierstrass process or self-similar martingales such as Lévy motion. One then forms the compound process

$$\mathcal{V}(t) := V(\mathcal{M}(t)). \tag{1.21}$$

To build an intuition let us recall the method of *midpoint displacement* which can be used to define simple Brownian motion $B_{1/2}$ which we will also call *Wiener motion* (WM) for a clear distinction from fBm. This method constructs $B_{1/2}$ iteratively at dyadic points. Having constructed $B_{1/2}(k2^{-n})$ and $B_{1/2}((k+1)2^{-n})$ one defines $B_{1/2}((2k+1)2^{-n-1})$ as $(B_{1/2}(k2^{-n}) + B_{1/2}((k+1)2^{-n}))/2 + X_{k,n}$. The off-sets $X_{k,n}$ are independent zero mean Gaussian variables with variance such as to satisfy (1.1) with $H = 1/2$. Thus the name of the method. One way to obtain *Wiener motion in multifractal time* WM(MF) is then to keep the off-set variables $X_{k,n}$ as they are but to apply them at the time instances $t_{k,n}$ defined by $t_{k,n} = \mathcal{M}^{-1}(k2^{-n})$, i.e., $\mathcal{M}(t_{k,n}) = k2^{-n}$:

$$\mathcal{B}_{1/2}(t_{2k+1,n+1}) := \frac{\mathcal{B}_{1/2}(t_{k,n}) + \mathcal{B}_{1/2}(t_{k+1,n})}{2} + X_{k,n}. \tag{1.22}$$

This amounts to a *randomly located random displacement*, the location being determined by $\mathcal{M}$. Indeed, (1.21) is nothing but a time warp.

An alternative construction of 'warped Wiener motion' WM(MF) which yields an equally spaced sampling as opposed to the samples $\mathcal{B}_{1/2}(t_{k,n})$ provided by (1.22) is desirable. To this end, note first that the increments of WM(MF) become independent Gaussians once the path of $\mathcal{M}(t)$ is realized. To be more precise, fix n and let

$$\mathcal{G}(k) := \mathcal{B}((k+1)2^{-n}) - \mathcal{B}(k2^{-n}) = B_{1/2}(\mathcal{M}(k+1)2^{-n})) - B_{1/2}(\mathcal{M}(k2^{-n})). \tag{1.23}$$

For a sample path of $\mathcal{G}$ one starts by producing first the random variables $\mathcal{M}(k2^{-n})$. Once this is done, the $\mathcal{G}(k)$ simply are independent zero-mean Gaussian variables with variance $|\mathcal{M}(k+1)2^{-n})) - \mathcal{M}(k2^{-n})|$. This procedure has been used in Figure 3.

Global analysis

For the right hand side (RHS) of the multifractal formalism (1.14), we need only to know that V is an H-sssi process, meaning that the increment $V(t+u) - V(t)$ is equal in distribution to $u^H V(1)$ (compare (1.1)). Assuming independence between V and $\mathcal{M}$ a simple calculation reads as

$$\begin{aligned} &\mathbb{E}_\Omega \sum_{k=0}^{2^n-1} |\mathcal{V}((k+1)2^{-n}) - \mathcal{V}(k2^{-n})|^q \\ &= \sum_{k=0}^{2^n-1} \mathbb{E}\mathbb{E}\left[|V(\mathcal{M}((k+1)2^{-n})) - V(\mathcal{M}(k2^{-n}))|^q \,\middle|\, \mathcal{M}(k2^{-n}), \mathcal{M}((k+1)2^{-n}) \right] \\ &= \sum_{k=0}^{2^n-1} \mathbb{E}\left[|\mathcal{M}((k+1)2^{-n}) - \mathcal{M}(k2^{-n})|^{qH} \right] \mathbb{E}\left[|V(1)|^q \right]. \end{aligned} \tag{1.24}$$

Here, we dealt with increments $|\mathcal{V}((k+1)2^{-n}) - \mathcal{V}(k2^{-n})|$ for the ease of notation. With little more effort they can be replaced by suprema, i.e., by $2^{-nh_k^{(n)}}$, or even by $2^{-nw_k^{(n)}}$ for certain wavelet coefficients and under appropriate assumptions (see theorem 8.5). It follows, e.g., for $h_k^{(n)}$, that

$$\text{Warped } H\text{-sssi:} \qquad T_{h,\mathcal{V}}(q) = \begin{cases} T_{h,\mathcal{M}}(qH) & \text{if } \mathbb{E}_\Omega \left[|\sup_{0 \le t \le 1} V(t)|^q \right] < \infty \\ -\infty & \text{else.} \end{cases} \tag{1.25}$$

Simple H-sssi process: When choosing the deterministic warp time $\mathcal{M}(t) = t$ we have $T_\mathcal{M}(q) = q - 1$ since $S_\mathcal{M}^{(n)}(q) = \text{const}2^n \cdot 2^{-nq}$ for all n. Also, $\mathcal{V} = V$. We obtain $T_\mathcal{M}(qH) = qH - 1$ which has to be inserted into (1.25) to obtain

$$\text{Simple } H\text{-sssi:} \qquad T_{h,V}(q) = \begin{cases} qH - 1 & \text{if } \mathbb{E}_\Omega \left[|\sup_{0 \le t \le 1} V(t)|^q \right] < \infty \\ -\infty & \text{else.} \end{cases} \tag{1.26}$$

Local analysis of warped fBm

Let us now turn to the special case where V is fBm. Then, we use the term FB(MF) to abbreviate *fractional Brownian motion in multifractal time*: $\mathcal{B}(t) = B_H(\mathcal{M}(t))$. First, to obtain an intuition on what to expect from the spectra of $\mathcal{B}$ let us note that the moments appearing in (1.25) are finite for all q as we will see in lemma 7.4. Applying the Legendre transform yields easily that

$$T^*_{\mathcal{B}}(a) = \inf_q (qa - T_{\mathcal{B}}(q)) = \inf_q (qa - T_{\mathcal{M}}(qH)) = T^*_{\mathcal{M}}(a/H), \qquad (1.27)$$

which is valid for all $a \in \mathbb{R}$ for which the second equality holds, i.e., for which the infimum is attained for q values in the range where $T_{\mathcal{B}}(q)$ is finite. In particular, for Brownian motion (fBm with $H = 1/2$) it holds for all a (compare lemma 7.4).

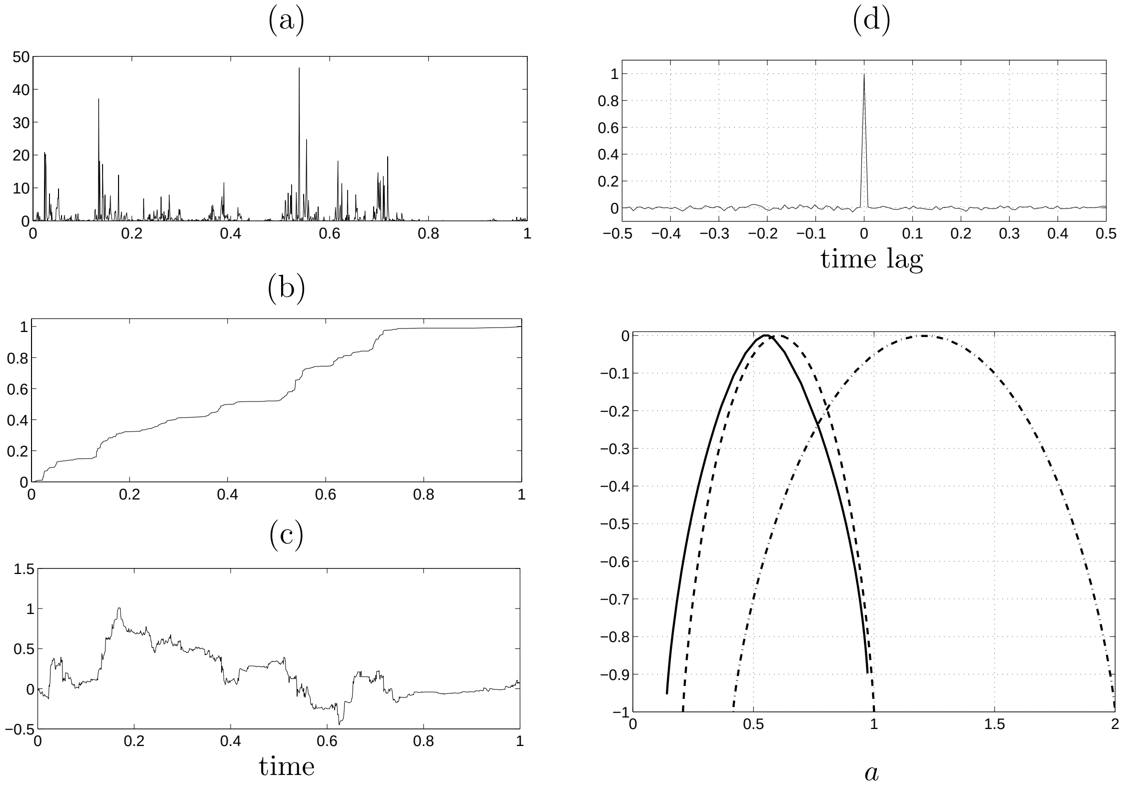

FIGURE 3. *Left: Simulation of Brownian motion in binomial time (a) Sampling of* $\mathcal{M}_b((k+1)2^{-n}) - \mathcal{M}_b(k2^{-n})$ $(k = 0,\ldots,2^n-1)$*, indicating distortion of dyadic time intervals (b)* $\mathcal{M}_b((k2^{-n}))$*: the time warp (c) Brownian motion warped with (b):* $\mathcal{B}(k2^{-n}) = B_{1/2}(\mathcal{M}_b(k2^{-n}))$
Right: Estimation of $\dim(E^{[a]}_{\mathcal{B}})$ *via* $\tau^*_{w,\mathcal{B}}$ *(d) Empirical correlation of the Haar wavelet coefficients. (e) Dot-dashed:* $T^*_{\mathcal{M}_b}$ *(from theory), dashed:* $T^*_{\mathcal{B}}(a) = T^*_{\mathcal{M}_b}(a/H)$ *Solid: the estimator* $\tau^*_{w,\mathcal{B}}$ *obtained from (c). (Reproduced from [40].)*

Second, towards the local analysis we recall the uniform and strict Hölder continuity of the paths of fBm. In theorem 7.3 we state a precise result due to Adler [5] which reads roughly as

$$\sup_{|u|\leq\delta} |\mathcal{B}(t+u)-\mathcal{B}(t)| = \sup_{|u|\leq\delta} |B_H(\mathcal{M}(t+u))-B_H(\mathcal{M}(t))| \simeq \sup_{|u|\leq\delta} |\mathcal{M}(t+u)-\mathcal{M}(t)|^H.$$

This is the key to conclude that B_H simply squeezes the Hölder regularity exponents by a factor H. Thus,

$$h_{\mathcal{B}}(t) = H \cdot h_{\mathcal{M}}(t), \qquad E_{\mathcal{M}}^{[a/H]} = E_{\mathcal{B}}^{[a]},$$

and, consequently, analogous to (1.27),

$$\boxed{\dim(E_{\mathcal{B}}^{[a]}) = \dim(E_{\mathcal{M}}^{[a/H]}).}$$

Figure 3 (d)-(e) displays an estimation of $\dim(E_{\mathcal{B}}^{[a]})$ using wavelets which agrees very closely with the form $\dim(E_{\mathcal{M}}^{[a/H]})$ predicted by theory. (For statistics on this estimator see [40, 41].)

Combining this with corollary 1.1 and (1.27) we may conclude:

Corollary 1.2 (Fractional Brownian Motion in Multifractal Time). *Let B_H denote fBm of Hurst parameter H. Let $\mathcal{M}(t)$ be of almost surely continuous paths and independent of B_H. Set $\mathcal{B}(t) = B_H(\mathcal{M}(t))$ and consider a multifractal analysis using $h_k^{(n)}$. Then, the* **multifractal warp formalism**

$$\boxed{\dim(E_{\mathcal{B}}^{[a]}) = f_{\mathcal{B}}(a) = \tau_{\mathcal{B}}^*(a) = T_{\mathcal{B}}^*(a) = T_{\mathcal{M}}^*(a/H)} \tag{1.28}$$

holds for any path and any a for which $\dim(E_{\mathcal{M}}^{[a/H]}) = T_{\mathcal{M}}^*(a/H) = T_{\mathcal{B}}^*(a)$.

The condition on a ensures that equality holds in the multifractal formalism for $\mathcal{M}$ and that the relevant moments are finite so that (1.27) holds. If satisfied, then the local, or fine, multifractal structure of $\mathcal{B}$ captured in $\dim(E_{\mathcal{B}}^{[a]})$ on the left side in (1.28) can be estimated through grain based, simpler and numerically more robust spectra on the right side, such as $\tau_{\mathcal{B}}^*(a)$ (compare Figure 3 (e)).

Moreover, the 'warp formula' (1.28) is appealing since it allows to *separate* the LRD parameter of fBm and the multifractal spectrum of the time change $\mathcal{M}$. Indeed, provided that $\mathcal{M}$ is almost surely increasing one has $T_{\mathcal{M}}(1) = 0$ since $S^{(n)}(0) = \mathcal{M}(1)$ for all n. Thus, $T_{\mathcal{B}}(1/H) = 0$ exposes the value of H. Alternatively, the tangent at $T_{\mathcal{B}}^*$ through the origin has slope $1/H$. Once H is known $T_{\mathcal{M}}^*$ follows easily from $T_{\mathcal{B}}^*$.

Simple fBm: When choosing the deterministic warp time $\mathcal{M}(t) = t$ we have $\mathcal{B} = B_H$ and $T_{B_H}(q) = qH - 1$ as in (1.26). In the special case of Brownian motion ($H = 1/2$) we may apply (1.28) for all a showing that all $h_k^{(n)}$-based spectra consist of the point $(H, 1)$ only. This makes the mono-fractal character of this process most explicit. In general, however, artifacts which are due mainly to diverging moments may distort this simple picture (see Section 7.3).

LRD and estimation of warped Brownian motion

Let $\mathcal{G}(k) := \mathcal{B}((k+1)2^{-n}) - \mathcal{B}(k2^{-n})$ be 'fGn in multifractal time' (see (1.23) for the case $H = 1/2$). Calculating auto-correlations explicitly, lemma 8.8 shows that $\mathcal{G}$ is second order stationary provided $\mathcal{M}$ has stationary increments. Assuming $\mathbb{E}[\mathcal{M}(s)^{2H}] = \text{const} \cdot s^{T(2H)+1}$, the correlation of $\mathcal{G}$ is of the form of ordinary fGn, but decaying as $r_{\mathcal{G}}(k) \simeq k^{2H_{\mathcal{G}}-2}$ where

$$H_{\mathcal{G}} = \frac{T_{\mathcal{M}}(2H) + 1}{2}. \tag{1.29}$$

Let us discuss some special cases. An obvious choice for a subordinator $\mathcal{M}$ is Lévy motion, an H'-self-similar, $1/H'$-stable process. It has independent, stationary increments. Since the relation (1.1) holds with H' as the scaling parameter, we have $T(q) = qH' - 1$ from (1.26). Moreover, $\mathcal{M}(s)^{2H}$ is equal in distribution to $(s^{H'}\mathcal{M}(1))^{2H}$ and indeed $\mathbb{E}[\mathcal{M}(s)^{2H}] = \text{const} \cdot s^{2HH'} = \text{const} \cdot s^{T(2H)+1}$. This expression is finite for $2H < 1/H'$. In summary, $H_{\mathcal{G}} = HH' < 1/2$.

For a continuous, increasing warp time $\mathcal{M}$, on the other hand, we have always $T_{\mathcal{M}}(0) = -1$ and $T_{\mathcal{M}}(1) = 0$. (Lévy motion is discontinuous; it is increasing for $H' < 1$, in which case $T(1)$ is not defined.) Exploiting the concave shape of $T_{\mathcal{M}}$ we find that $H < H_{\mathcal{G}} < 1/2$ for $0 < H < 1/2$, and $1/2 < H_{\mathcal{G}} < H$ for the LRD case $1/2 < H < 1$.

Especially in the case of $H = 1/2$ ('white noise in multifractal time') $\mathcal{G}(k)$ becomes *uncorrelated* (see also (8.20)). Notably, this is a stronger statement than the observation that the $\mathcal{G}(k)$ are *independent conditioned* on $\mathcal{M}$ (compare Section 1.6). As a particular consequence, wavelet coefficients will decorrelate fast for the compound process $\mathcal{G}$, not only when conditioning on $\mathcal{M}$ (compare Figure 3 (d)). This is favorable for estimation purposes as it reduces the error variance. Finally, for increasing $\mathcal{M}$ we have $T(1) = 0$ and the requirements for (1.29) reduce to the simple $\mathbb{E}[\mathcal{M}(s)] = s$. Multiplicative processes with this property (as well as stationary increments) have been introduced recently [14, 70, 74, 105].

Though seemingly obvious it should be pointed out that the vanishing correlations of $\mathcal{G}$ in the case $H = 1/2$ should not be taken as an indication of independence. After all, $\mathcal{G}$ becomes Gaussian only when conditioning on knowing $\mathcal{M}$. A strong, higher order dependence in $\mathcal{G}$ is hidden in the dependence of the increments of $\mathcal{M}$ which determine the variance of $\mathcal{G}(k)$ as in (1.23). Indeed, Figure 3 (c) shows clear phases of monotony of $\mathcal{B}$ indicating positive dependence in its increments $\mathcal{G}$, despite vanishing correlations. Mandelbrot calls this the 'blind spot of spectral analysis'.

Multifractals in multifractal time

Despite of its simplicity the presented scheme for constructing multifractal processes allows for various play-forms some of which are little explored. First of all, for simulation purposes one might subject the *randomized Weierstrass-Mandelbrot function* to time change rather than fBm itself.

Next, we may choose to replace fBm with a more general self-similar process (7.1) such as Lévy motion. Difficulties arise here since Lévy motion is itself a multifractal with varying Hölder regularity and the challenge lies in studying which exponents of the 'multifractal time' and the motion are most likely to meet. A solution for the spectrum $f(a)$ is given in corollary 8.13 which actually applies to arbitrary processes Y with stationary increments (compare theorem 8.15) replacing fBm. In its most compact form our result reads as:

Corollary 1.3 (Lévy motion in multifractal time). *Let L_H denote Lévy stable motion and let $\mathcal{M}$ be a binomial cascade (see 5.1) independent of L_H and set $\mathcal{V}(t) = L_H(\mathcal{M}(t))$. Then, for almost all paths*

$$f_{\mathcal{V}}(a) = \tau^*_{\mathcal{V}}(a) \overset{\text{a.s.}}{=} T^*_{\mathcal{V}}(a) \tag{1.30}$$

for all α where $T^*_{\mathcal{V}} > 0$. The envelope $T^*_{\mathcal{V}}$ can be computed through the **warp formula**

$$\boxed{T_{\mathcal{V}}(q) = T_{\mathcal{M}}\Big(T_{L_H}(q) + 1\Big)\ .} \tag{1.31}$$

Recall (1.26) for a formula of T_{L_H}, which is generalized in (7.10). As the paper shows (1.30) and (1.31) hold actually in more generality.

Finally, for $\mathcal{Y}(t) = Y(\mathcal{M}(t))$ where Y and $\mathcal{M}$ are both almost surely increasing, i.e., multifractals in the classical sense, a close connection to the so-called 'relative multifractal analysis' [95] can be established using the concept of inverse multifractals [94]: understanding the multifractal structure of $\mathcal{Y}$ is equivalent to knowing the multifractal spectra of Y with respect to $\mathcal{M}^{-1}$, the inverse function of $\mathcal{M}$. We will show how this can be resolved in the simple case of binomial cascades.

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