Multifractals and Wavelets: A potential tool in Geophysics

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SEG meeting 1998, Technical Session on Novel transforms, New Orleans, LA, September 1998

Summary

The study of fractal quantities and structures exhibiting highly erratic features on all scales has proved to be of outstanding significance in various disciplines. While scaling phenomena are pervasive in natural and man-made constructs, such objects are less fractal than multifractal. In most simple terms this means that moments of different orders scale differently with increasing resolution.

This paper should be understood as a tutorial in multifractals and their analysis via wavelets, in view of possible applications in geophysics. It is elaborated how a description of the well log measurement through wavelets provides a new way of modeling reflection of waves in a material which is dependent on frequency. The wavelet analysis has the potential to provide an explanation for the inconstencies that are observed when comparing subsurface models that have been constructed from measurements with different resolutions, such as surface seismic, vertical seismic profiles and well logs.

Introduction

Multifractal structures have been found in various contexts, most prominently in the study of turbulence, stock market exchange rates and recently also network data traffic, introducing fruitful and novel aspects to the mentioned fields. This paper is written in the hope of achieving the same in the area of geophysics.

The idea of using multifractal measures in geophysics is not new, but has been pioneered by Mandelbrot (Mandelbrot, 1989). More recent work (Herrmann, 1997) attempts to acommodate the concept of a multifractal subsurface in the theory of seismic wave propagation.

Fractals are objects of a complex structure on all scales. Here, we are mostly interested in functions and processes. Fractal functions, e.g., are nowhere differentiable (ubiquitous details when zooming in), and fractal processes do not show convergence in the sense of a central limit theorem (zooming out). A useful parameter for the complexity of such fractal sets is the fractal dimension, which is closely related to the degree of Hölder regularity which again can be thought of generalizing the degree of differentiability to real numbers.

The fractal dimension, however, is a global parameter which measures the 'overall worst' behavior and does not account for a possible variability of the degree of regularity. Multifractal analysis, thus, ideally aims towards a compact representation of the 'spectral decomposition' of a signal into parts of equal strength of regularity.

While it is possible to provide such a representation in different ways, the most convenient in the present context is using a the wavelet transform. This analytical tool can be thought of as testing the signal with a waveform which is localized in time (or space) and in frequency. It has, thus, the advantage over the Fourier transform of giving information not only of the global frequency content of a signal but also shows where in time certain frequency components occur. Wavelets, being constructed through rescaling of a mother wavelet, provide a natural tool for a scaling analysis.

Well log data, such as sonic velocity measurements, often clearly show a multifractal structure (Herrmann, 1997). So, first of all multifractal models have an application in the construction of realistic models of the subsurface (Mandelbrot, 1989) (e.g. velocity models).

An issue of central importance in exploration seismology is the connection of well log measurements to seismic data. These two types of measurements clearly take place on a different scale, and hence the scaling properties of the subsurface should be taken into account when comparing seismic and well log data. The multifractal character of well logs indicates that an understanding of wave propagation through a multifractal structure may be an important step towards the incorporation of measurement scale into our seismic models. In this paper, we challenge the classically trained mind with a novel point of view, which is to abandon the usual picture of a subsurface that consists of a stack of homogeneous layers, but to view it rather as a material parts of which will act as reflectors depending on frequency (or scale).

Being meant as a tutorial, this paper introduces the principles of multifractals and explains how wavelets are of use as a tool for multifractal analysis.

Multifractal Analysis

A nice, more elaborate introduction to the basic ideas of multifractal analysis and an extensive bibliography can be found in the introductory parts of (Riedi, 1995; Riedi, 1996; Riedi and Mandelbrot, 1998), as well as in (Evertsz and Mandelbrot, 1992). In this paper we are limited to presenting only what is important in this context.

Let us start by noting the conceptually important fact that a 'fractal' is a set, while a multifractal is a measure, or distribution. In other words, while fractal geometry is interested in the complexity of a set which is measured by one single parameter, the dimension, multifractal analysis is committed to account for inhomogeneities.

Multifractals and Wavelets

Fractals

The study of fractal quantities and structures has proved to be of outstanding significance in various disciplines. As one of the most prominent examples, fractional Brownian motion (fBm) has a 'fractal' or highly erratic appearance which is intimately related to its spectral properties. Indeed, it is not surprising that non-differentiability is reflected in frequency representations. Consequently, the most efficient estimators of the fractal parameter H of fBm are using the power spectrum, and wavelets.

However, fBm is not a typical fractal object since it is homogeneous, or monofractal, i.e. its local degree of Hölder continuity H_t is the same at all times t. On the other hand, most real world signals and an increasing number of processes have been shown to exhibit multifractal structure, meaning that H_t takes different values and actually varies erratically in time. It is clear that such a structure calls for a time dependent frequency analysis. Therefore, wavelets are the optimal tool, being well localized both in time and in frequency.

Singularities

A function or a process $Y(t)$ is called $C^\alpha(t_0)$ if there is a polynomial P and a constant C such that

$$|Y(t) - P(t)| \leq C \cdot |t - t_0|^\alpha \qquad (1)$$

for t close to t_0. The local degree of Hölder continuity H_{t_0} at t_0 is then the largest $\alpha > 0$ for which Y is $C^\alpha(t_0)$.

It is clear that such a notion is mainly of theoretical importance and hard to deal with in real world estimations. Wavelets, as mentioned before, seem to have the most potential to approach such a task. We postpone elaborating on these issues to the end, though, and explain the object of multifractal analysis in a more narrow context, which is actually completely sufficient for the scope of this paper.

Having analysis and modeling of well log data in mind we may concentrate on processes Y with positive increments X. Often, it is in fact this positive process X we are interested in, such as a velocity or density field. But as will become apparent in an instant it is more practical to construct Y.

If we further assume that the data has no polynomial trends, i.e. P in (1) reduces to the constant $Y(t_0)$, the definition of H_t simplifies enormously to

$$H_t = \lim_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \log |Y(t+\varepsilon) - Y(t-\varepsilon)|. \qquad (2)$$

(If the indicated limit does not exist for a t we will replace it by the liminf.)

Multiplicative Cascades

The most simple example of a positive increment process with multifractal properties are multiplicative cascades. Introduced by Mandelbrot (Mandelbrot, 1974) as a model of energy dissipation of the velocity field in turbulence they are well understood by now (Kahane and Peyrière, 1976; Holley and Waymire, 1992; Riedi, 1995).

For simplicity we will construct the process Y on the time interval $I := [0,1]$ such that $Y(0) = 0$. To start, set $Y(1)$ equal to some positive random value M_0^0. The observation that M_0^0 is the increment of the limiting process Y over I, though trivial, will help in the sequel.

To get the iterative construction going, choose 2 positive random variables M_0 and M_1 over some probability space Ω such that $M_0 + M_1 = 1$ for almost all $\omega \in \Omega$. Denote the two halves of I by $I_0^1 = [0, 1/2]$ and $I_1^1 = [1/2, 1]$, and define $M_i^1 \cdot M_0^0$ to be the increment of Y over I_i^1, where M_i^1 is distributed as M_i. Due to $M_0 + M_1 = 1$, this is in agreement with the first step of the construction.

Now iterate replacing I by I_i^1. At stage n, the total increment M_0^0 over I is split over the 2^n dyadic intervals of order n. To be more precise, let us use the notation $k_i := 2k_{i-1} + k_i'$ with $k_i' = 0$ or 1, starting iteration with $k_{-1} := 0$. In other words, $k_n 2^{-n} = \sum_{i=0}^{n} k_i' 2^{-i}$ lies in $[0,1]$ and has binary digits k_i'. Since $k_{n+1} \operatorname{div} 2 = k_n$ and $k_{n+1} \bmod 2 = k_n'$, the increment of Y over $I_{k_n}^n := [k_n 2^{-n}, (k_n+1)2^{-n}]$ is $M_{k_n}^n \cdot M_{k_{n-1}}^{n-1} \cdots M_{k_1}^1 \cdot M_0^0$, where the random variables $M_{k_n}^n$ i.i.d., with law $M_{k_n'}$.

While we allow dependence within scale, that is, among $M_{k_n}^n$ for fixed n but different choices of k_i', we enforce independence across scale, that is, between $M_{k_n}^n$ for fixed digits k_i' $(i = 0, 1, 2, \ldots)$ and different n.

Iterating indefinitely, we will have defined Y in the dyadic points. Since the multipliers are smaller than 1 the increments over dyadic intervals tend to zero. Due to our requirement that Y be increasing, it must be continuous and the whole process is defined. It is now clear that Y is much more compact model than the sequence of dyadic increment processes X^n. A derivative of Y in the usual sense, the other option of a compact representation, does not exist as follows from (5) below.

Singularities of Cascades

To understand the multifractal properties of this binomial cascade process Y let us assume for the moment that the multipliers were chosen deterministically, i.e. $M_{k_n}^n = m_{k_n'}$ for some fixed numbers m_0 and m_1. Also, for simplicity let us focus on dyadic increments, i.e. $\delta = 2^{-n}$. If t has binary digits k_i' in the notation introduced above, then the increment over the dyadic interval containing t is $X_{k_n}^n = Y((k_n+1)2^{-n}) - Y(k_n 2^{-n})$. Being a product of the multipliers $M_{k_i}^i$, this can then be written in a simple manner using the number of zeros $l(n)$ among the first n dyadic digits k_i' of t:

$$X_{k_n}^n = m_0^{l(n)} m_1^{n-l(n)}. \qquad (3)$$

So, if the limiting frequency of digits $\phi = \lim_n l(n)/n$

exists, H_t takes the form

$$H_t = \lim_{n\to\infty} (1/n)\log_2 X^n_{k_n} = \phi\log_2 m_0 + 1 - \phi\log_2 m_1. \tag{4}$$

Multifractal structure of Cascades

It follows immediately from (4) that we will find the whole interval between $\log_2 m_0$ and $\log_2 m_1$ as values H_t. The multifractal spectrum will now express how frequently, or how rarely these values will occur. To obtain such information we best employ limiting theorems from probability theory, where we consider an adequate random choice of t. (Recall that Y is still deterministic.)

First, choosing t uniformly, the Law of Large Numbers (LLN) shows that $\phi = 1/2$ 'almost surely', i.e. for a set of (Lebesgue) mass 1 we have $H_t = a_0 := (1/2)\log_2(m_0 m_1)$. Since we can have a whole range of H_t, $l(n)/n$ must approximate values different from the expected a_0. Due to the LLN, their t-probability (or relative number) will have to decay to zero as n increases. Theorems on Large Deviation Principles (LDP) state that this decay is exponentially fast. Using (3) one finds easily, as $n\to\infty$ and $\varepsilon\to 0$ (Riedi, 1995; Riedi, 1996):

$$P_t[a-\varepsilon \le (1/n)\log_2|X^n_k| \le a+\varepsilon] \simeq 2^{n(f(a)-1)} \tag{5}$$

We take this rate function $f(a)$ as our measurement of 'how rarely' one will find $H_t = a$, and call it *multifractal spectrum*. Since $P_t[\ldots] = (1/2^n)\#\{k\ldots\}$, (5) expresses that the 'normalized logarithmic histograms' $(1/n)\log P_t[\ldots]$ should collapse onto one curve f in the limit $n\to\infty$. They form, thus, a 'scale invariant'.

It follows from (4) that Y is differentiable with derivative 0 in all points t with $H_t > 1$. Since $a_0 > 1$ this will be the case almost everywhere. In points t with $H_t < 1$, on the other hand, a 'derivative' would take the value ∞. In other words, the increment process X^n will have to provide the bulk of the innovations of Y in fewer time intervals as n grows. In the limit it must, therefore, be singular and a derivative of Y exists only in the distributional sense.

Multifractal formalism

Precisely speaking, (5) is a statement about observing approximative Hölder continuity on coarse scale, rather than H_t itself. However, in the given context it can be shown that f has also the desired geometric meaning, i.e. it gives the fractal dimension of the set of points t with $H_t = a$ (Riedi, 1995; Cawley and Mauldin, 1992). The smaller this dimension is, the 'thinner' the corresponding set.

In practice, dimensions are difficult to estimate whence the definition (5) of the spectrum f is more suitable for application. Moreover, the theory of LDP provides us with a numerically more robust tool for an estimation of f, the moment generating function τ of t-random variables $\log|X^n_{k_n}|$ which involves averaging: Using $\mathbb{E}_t[\exp(q\cdot\log|X^n_{k_n}|)] = 2^{-n}\sum_{k=0}^{2^n-1}|X^n_k|^q$ one can write

$$\tau(q) := \lim_{n\to\infty}(1/n)\log_2\sum_{k=0}^{2^n-1}|X^n_k|^q \tag{6}$$

In practice, one will estimate $\tau(q)$ as the least square fitting slope of $\log_2\sum|X^n_k|^q$ against n. The multifractal formalism reads as (Riedi, 1995; Riedi, 1998):

$$\tau(q) = f^*(q) := \inf_a (qa - f(a)),$$

i.e. $\tau(q) = af'(a) - f(a)$ at $q = f'(a)$.

Consequently, $f(a) = q\tau'(q) - \tau(q)$ at $a = \tau'(q)$, but in general $f(a) \le \tau^*(a)$. As the Legendre transform of f, $\tau(q)$ is a concave function, and the same can be expected from f in nice cases, e.g. when $\tau(q)$ is differentiable for all q. As an example let us compute the cascade above:

$$\tau(q) = \lim_{n\to\infty}(1/n)\log_2({m_0}^q + {m_1}^q)^n = \log_2({m_0}^q + {m_1}^q)$$

Random Cascades

Having dealt with deterministic setting, let us return to random signals. The assumption of independence of the multipliers will allow to use martingale arguments to show convergence of the process. While a more complete account on a generalization of (5) can be found in (Riedi, 1998) we can mention here only how to generalize (6): One sets simply

$$\tau(q) := \lim_{n\to\infty}(1/n)\log_2\mathbb{E}\sum_{k=0}^{2^n-1}|X^n_k|^q$$

replacing the expectation over time t by the one over t and ω.

For the random cascade, one finds

$$\tau(q) = \log_2(\mathbb{E}[{M_0}^q] + \mathbb{E}[{M_1}^q])$$

As was shown in (Kahane and Peyrière, 1976) this function carries important information besides the multifractal spectrum. One fact of interest might be, that the increment process will have marginals with diverging moments of all orders $q > 1$ such that $\tau(q)$ is negative. This provides a useful alternative to stable processes which are hard to simulate.

Wavelets

In our exposition so far the simplification (2) of local regularity was essential. When dealing with polynomial trends as well as with more general functions – not necessarily of positive increments – wavelets, or a more general time-frequency analysis is the tool of choice to detect multifractal structures.

Multifractals and Wavelets

In very rough terms, a wavelet ψ is a function which is well localized in space and frequency. Preferably, ψ would have compact support and its Fourier transform should decay fast. Furthermore, ψ should meet some regularity conditions which can be found in any related text (Mallat, 1989; Daubechies, 1992). Using rescaled and dislocated versions $\psi_{n,k}(t) := \psi(2^n(t-k))$ one will obtain an L^2 basis.

Since the frequency content of $\psi_{n,k}$ is shifted depending on n, the decomposition of a signal Y in this basis will then provide valuable information about the location of oscillations and, thus, singularities. As a matter of fact the connection between Hölder continuity (1) and the decay of wavelet coefficients $c_{n,k} := \int Y\psi_{n,k}$ can be made rather precise using Besov spaces and Sobolov embedding (Jaffard, 1991).

To be more precise, one property which makes wavelet transforms attractive is the fact that they provide a compact representation. As has been observed in many contexts, most coefficients are very small, the large one lying on so-called 'lines of maxima'. These lines bifurcate as resolution increases, reflecting in some way the multifractal structure of the signal. For certain multiplicative cascades similar as introduced above it has been shown that replacing the increments X_k^n in (6) by the $c_{n,k}$ on the lines of maxima will allow to estimate the multifractal spectrum (Bacry et al., 1993; Jaffard, 1993).

Conclusions

In this paper we propose a novel way of putting the wavelet transformation to use in exploration seismology. It has been demonstrated convincingly by (Herrmann, 1997) that well log data shows multifractal structure. At a given scale n, or equivalently for a given frequency, the location of the lines of maxima of the wavelet transform of the impedance log could indicate the location where a wave-pulse of the given frequency would experience its most important reflection. Since these locations are scale dependent we suggest to view the subsurface as a material which will reflect waves depending on their frequency content.

Acknowledgments

The author expresses his sincere gratitude to Philippe Steeghs for his indispensable help in the preparation of this paper. He would also like to thank Jacob Fokkema and Felix Herrmann for inspiring discussions. Support of the author comes from NSF grant no. MIP–9457438, and by the Office of Naval Research, grant no. N00014–95–1–0849.

References

Bacry, E., Muzy, J., and Arneodo, A., 1993, Singularity spectrum of fractal signals from wavelet analysis: Exact results: J. Stat. Phys., **70**, 635–674.

Cawley, R., and Mauldin, R. D., 1992, Multifractal decompositions of moran fractals: Advances Math., **92**, 196–236.

Daubechies, I., 1992, Ten lectures on wavelets: SIAM, New York.

Evertsz, C. J. G., and Mandelbrot, B. B., 1992, Multifractal measures: Appendix B in: 'Chaos and Fractals' by H.-O. Peitgen, H. Jürgens and D. Saupe, Springer New York, pages 849–881.

Herrmann, F., 1997, A scaling medium representation: A discussion on well-logs, Fractals and Waves: Ph.D. thesis, Delft University of Technology, Netherlands.

Holley, R., and Waymire, E., 1992, Multifractal dimensions and scaling exponents for strongly bounded random cascades: Ann. Appl. Prob., **2**, 819–845.

Jaffard, S., 1991, Pointwise smoothness, two-microlocalization and wavelet coefficients: Publicacions Matematiques, **35**, 155–168.

Jaffard, S., 1993, Multifractal formalism for functions: CRAS, **317**, 745–750.

Kahane, J.-P., and Peyrière, J., 1976, Sur certaines martingales de Benoit Mandelbrot: Adv. Math., **22**, 131–145.

Mallat, S. G., Sept. 1989, Multiresolution approximations and wavelet orthonormal bases of $L^2(R)$: Trans. Amer. Math. Soc., **315**, no. 1, 69–87.

Mandelbrot, B. B., 1974, Intermittent turbulence in self similar cascades : divergence of high moments and dimension of the carrier: J. Fluid. Mech., **62**, 331.

Mandelbrot, B. B., 1989, Multifractal measures, especially for the geophysicist: PAGEOPH, **131**, 5–42.

Riedi, R. H., and Mandelbrot, B. B., 1998, Exceptions to the multifractal formalism for discontinuous measures: Math. Proc. Cambr. Phil. Soc., **123**, 133–157.

Riedi, R. H., 1995, An improved multifractal formalism and self-similar measures: J. Math. Anal. Appl., **189**, 462–490.

Riedi, R., 1996, Multifractals: An introduction: Research report, Rice University 1997, `http://www.dsp.rice.edu/~riedi`.

Riedi, R. H., 1998, Multifractal processes: Prob. Th. Rel. Fields, submitted.