*Math. Proc. Camb. Phil. Soc. 123 (1998) 133–157*
*printed in Great Britain*

# Exceptions to the multifractal formalism for discontinuous measures

BY RUDOLF H. RIEDI
*INRIA Rocquencourt, B.P. 105*
*78153 Le Chesnay Cedex, France*
*email:* Rolf.Riedi@inria.fr, riedi@rice.edu

AND BENOIT B. MANDELBROT
*Mathematics Department, Yale University,*
*New Haven CT 06520-8283, USA*
*email:* fractal@watson.ibm.com

(*Received* 14 *August* 1995 ; *revised* 29 *April* 1996)

**Abstract**

In an earlier paper [MR] the authors introduced the *inverse measure* $\mu^\dagger(dt)$ of a given measure $\mu(dt)$ on $[0,1]$ and presented the 'inversion formula' $f^\dagger(\alpha) = \alpha f(1/\alpha)$ which was argued to link the respective multifractal spectra of $\mu$ and $\mu^\dagger$. A second paper [RM2] established the formula under the assumption that $\mu$ and $\mu^\dagger$ are continuous measures.

Here, we investigate the general case which reveals telling details of interest to the full understanding of multifractals. Subjecting self-similar measures to the operation $\mu \mapsto \mu^\dagger$ creates a new class of discontinuous multifractals. Calculating explicitly we find that the inversion formula holds only for the 'fine multifractal spectra' and not for the 'coarse' ones. As a consequence, the multifractal formalism fails for this class of measures. A natural explanation is found when drawing parallels to equilibrium measures. In the context of our work it becomes natural to consider the degenerate Hölder exponents 0 and $\infty$. (fac simile for personal use ©Cambridge Philosophical Society)

## 1 Introduction

Let $\mu$ be a probability measure on $[0,1]$. Its distribution function $M(x) = \mu([0,x])$ is an increasing and right-continuous map of $[0,1]$ to itself. There is a natural way of defining an 'inverse function' $M^\dagger$ of $M$. Its differential is a probability measure $\mu^\dagger$ which we call the *inverse measure* of $\mu$:

$$\mu^\dagger([0,\theta]) := M^\dagger(\theta) := \begin{cases} \inf\{t \,:\, M(t) > \theta\} & \text{if } \theta < 1 \\ 1 & \text{if } \theta = 1. \end{cases} \tag{1}$$

As will be shown $\mu^\dagger$ is indeed a measure, and $\mu^{\dagger\dagger} = \mu$.

Our interest lies in a possible relation between the multifractal spectra $f$ and $f^\dagger$ of $\mu$ and $\mu^\dagger$ and the implications of such a connection. (For definitions see Section 2). In part I [MR] it was argued that the so-called *inversion formula* should hold:

$$f^\dagger(\alpha) = \alpha f(1/\alpha). \tag{2}$$

Part II [RM2] established the formula under the assumption that $\mu$ and $\mu^\dagger$ are continuous.

The practical use of such a formula is most evident when dealing with left-sided spectra [M90, MEH, RM1] since it allows to transform the infinite range $[\alpha_{\min}, \infty]$ of Hölder exponents of a left-sided spectrum into the finite range $[0, 1/\alpha_{\min}]$ of a right-sided spectrum.

A further application of the inversion formula is to self-similar measures which reveals telling details on the multifractal formalism. Recall that a compactly supported measure $\mu$ is traditionally called *self-similar* iff

$$\mu = \sum_{i=0}^{u-1} p_i \mu(w_i^{-1}(\cdot)), \tag{3}$$

where $w_0, \ldots, w_{u-1}$ are similarity maps of $\mathbb{R}^d$ with *contraction ratios* $r_i \in (0,1)$, and where the *probabilities* $p_i > 0$ satisfy $p_0 + \ldots + p_{u-1} = 1$. As Hutchinson [H] showed, such measures exist and are unique even under the weaker condition that the $w_i$ are contractions.

Computation of the multifractal spectrum requires knowledge on the amount of possible overlap in (3). The widely used *open set condition OSC* of Hutchinson [H] is said to hold if there is a bounded, open set $O$ such that $w_i(O)$ are mutually disjoint subsets of $O$. For the ease of dealing with inverse measures of self-similar measures, we will assume that the OSC holds with $O = (0,1)$. Then, it is well-known (see [AP, R1] and also [CM, F2, O]) that all reasonable definitions of the multifractal spectrum of $\mu$ coincide. In particular, all spectra equal the Legendre transform $\beta^*(\alpha) := \inf_q(q\alpha - \beta(q))$ where

$$\sum_{i=0}^{u-1} {p_i}^q {r_i}^{-\beta(q)} = 1. \tag{4}$$

It is easy enough to verify the inversion formula (2) for self-similar measures with full support $[0,1]$: In this case we have $r_0 + \ldots + r_{u-1} = 1$, and a moments thought shows that the inverse measure $\mu^\dagger$ is self-similar with ratios $r_i^\dagger = p_i$, and probabilities $p_i^\dagger = r_i$. Thus, $q = -\beta^\dagger(q^\dagger)$, $q^\dagger = -\beta(q)$, and (2) follows easily from $f = \beta^*$.

If $\mu$ is supported on a Cantor set $K \subset [0,1]$ then $r_0 + \ldots + r_{u-1} < 1$ by the OSC (note that $\dim(K) = -\beta(0) < 1$). In order to obtain an invariance for $\mu^\dagger$ it is useful to add similarities $w_j$ $(j = u, \ldots v-1)$ to the family $w_0, \ldots, w_{u-1}$ such that $(0,1)$ is still an open set and such that $r_0 + \ldots + r_{v-1} = 1$. Assigning the probabilities $p_j = 0$ $(j = u, \ldots v-1)$ to these maps leaves $\mu$ unchanged and finds $\mu^\dagger$ invariant under $(w_0^\dagger, \ldots, w_{u-1}^\dagger)$.

This observation leads naturally to extending the notion of self-similar measures by allowing ratios $r_i = 0$ and probabilities $p_i = 0$. A first possible extension of the inversion formula for non-continuous measures is, thus, to verify whether (4) (the sum taken only over all $i$ with $r_i^\dagger \neq 0$, i.e. $p_i \neq 0$) continues to rule the spectra of this broader class of self-similar measures. As we will show, this is indeed true for the two Hölder spectra $f_{\mathrm{H}}(\alpha)$ and $f_{\mathrm{P}}(\alpha)$ which are defined as the Hausdorff and the packing dimension of the set $K_\alpha$ of singularity exponents $\alpha$, respectively (see Section 2). The 'coarse' Hölder spectra

$f_{\mathrm{G}}(\alpha)$ and $f_{\mathrm{L}}(\alpha)$, however, which are obtained through partitioning of $[0,1]$, contain less information on the singularities than $f_{\mathrm{H}}$, and the inversion formula fails here. This is due to the presence of atoms. They shadow the finer details of the dense parts of the measure to an analysis from the 'global' point of view of $f_{\mathrm{G}}$, which manifests itself in a linear part in the graph of $f_{\mathrm{G}}$.

As a consequence, the multifractal formalism which states $f_{\mathrm{H}} = f_{\mathrm{P}} = f_{\mathrm{G}} = f_{\mathrm{L}}$, fails for this class of multiplicative measures. Moreover, the inversion formula (2) does not hold for $f_{\mathrm{G}}$ and $f_{\mathrm{L}}$ in general. In more positive words, information hidden in a linear part of $f_{\mathrm{G}}$ may be recovered by analyzing its inverse measure. It is worthwhile to note that such a procedure is not equivalent to the 'fixed mass algorithm', unless $\mu$ is continuous and non-vanishing.

Section 2 provides definitions and the proof of (2) in the continuous case. In Section 3 the discontinuous self-similar measures are introduced and their full multifractal analysis is provided. Section 4 contains the proof of (2) for $f_{\mathrm{H}}$ and $f_{\mathrm{P}}$ for general probability measures on $[0,1]$..

# 2 Preliminaries

We start this section by establishing some claims made in the introduction. Then, we introduce the various multifractal spectra and relate them to each other. Finally, we prove the inversion formula (2) in the continuous case.

**Lemma 1** *$M^\dagger$ as defined in (1) is monotonous and right-continuous. Hence, $\mu^\dagger$ is a measure.*

**Proof**
Monotony of $M^\dagger$ is immediate. Consider a sequence $\theta_k \searrow \theta$. By definition of $M^\dagger(\theta)$, we can choose $\{t_n\}_n$ such that $M(t_n) > \theta$ and $t_n < M^\dagger(\theta) + 1/n$. For every $n$ we find $k_n$ with $M(t_n) > \theta_{k_n}$, hence, $t_n \geq M^\dagger(\theta_{k_n}) \geq M^\dagger(\theta)$ and $M^\dagger$ is right-continuous. $\diamondsuit$

**Lemma 2** *We have ${\mu^\dagger}^\dagger = \mu$. In other words, ${M^\dagger}^\dagger = M$.*

**Proof**
Take $t < 1$ and let $\theta := M(t)$. Recall that ${M^\dagger}^\dagger(t) = \inf\{\theta' : M^\dagger(\theta') > t\}$.
Assume first that ${M^\dagger}^\dagger(t) < \theta$. Then, we find $\theta' < \theta$ with $M^\dagger(\theta') > t$. Take $t' > t$ with $M^\dagger(\theta') > t'$. The definition of $M^\dagger$ implies $M(t') \leq \theta' < \theta = M(t)$, a contradiction to monotony.
Assume now that ${M^\dagger}^\dagger(t) > \theta$. Then, we find $\theta' > \theta$ with $M^\dagger(\theta') \leq t$. Take $t' > t$. The definition of $M^\dagger$ implies $M(t') > \theta'$. Letting $t' \searrow t$ yields $M(t+) \geq \theta' > \theta$, a contradiction to right-continuity. $\diamondsuit$

## 2.1 The multifractal formalism

Recall the definition of *$\gamma$-dimensional Hausdorff measure* in $\mathbb{R}^d$

$$\eta^\gamma(E) = \sup_{\delta \to 0} \eta^\gamma_\delta(E), \qquad \eta^\gamma_\delta(E) = \inf\Big\{ \sum_k |I_k|^\gamma : E \subset \cup_k I_k \text{ and } |I_k| \leq \delta \Big\}$$

where $|I|$ stands for the diameter of $I$ and where the sets $I_k$ are arbitrary. The *Hausdorff dimension* is then defined as

$$\dim(E) = \inf\{\gamma \geq 0 \,:\, \eta^\gamma(E) = 0\} = \sup\{\gamma \geq 0 \,:\, \eta^\gamma(E) = \infty\}.$$

Following Tricot [Tr], we define the *$\gamma$-dimensional packing pre-measure*

$$\hat\pi^\gamma(E) = \inf_{\delta\to 0} \hat\pi^\gamma_\delta(E), \qquad \hat\pi^\gamma_\delta(E) = \sup\Big\{\sum_k |I_k|^\gamma \,:\, \{I_k\}_k \text{ is a } \delta\text{-packing of } E\Big\}$$

A $\delta$-packing $\{I_k\}_k$ of $E$ is a collection of mutually disjoint, open balls, each of length at most $\delta$ and each intersecting $E$. The *$\gamma$-dimensional packing measure* is given by

$$\pi^\gamma(E) := \inf\left\{\sum_n \hat\pi^\gamma(E_n) \,:\, E \subset \textstyle\bigcup_n E_n\right\}$$

(the sets $E_n$ are arbitrary) and the *packing dimension* by

$$\mathrm{Dim}(E) = \inf\{\gamma \geq 0 \,:\, \pi^\gamma(E) = 0\} = \sup\{\gamma \geq 0 \,:\, \pi^\gamma(E) = \infty\}.$$

For convenience, we set $\dim(\emptyset) = \mathrm{Dim}(\emptyset) = -\infty$. Let $\mu$ be a measure on $[0,1]^d$. Given a number $\alpha$, $0 \leq \alpha \leq \infty$, called 'Hölder exponent', set

$$\begin{aligned} F_\alpha &= \left\{t \in [0,1]^d \,:\, \limsup_{I\to\{t\}} \frac{\log \mu(I)}{\log |I|} \leq \alpha\right\} \\ G_\alpha &= \left\{t \in [0,1]^d \,:\, \liminf_{I\to\{t\}} \frac{\log \mu(I)}{\log |I|} \geq \alpha\right\} \end{aligned}$$

with the convention $\log 0 = -\infty$. Here, $I \to \{x\}$ means that $I$ is a cube containing $x$, and that the length of $I$ tends to zero. Finally, set

$$\begin{aligned} K_{\alpha,\alpha'} &= G_\alpha \cap F_{\alpha'} \\ K_\alpha &= K_{\alpha,\alpha}. \end{aligned}$$

$K_\alpha$ is sometimes called the 'set of Hölder exponent $\alpha$'. Denote the corresponding sets of $\mu^\dagger$ by $F^\dagger_\alpha$ etc.

**Definition 3** *The two* fine multifractal spectra *are the* Hausdorff spectrum *and the* packing spectrum*, respectively, which are defined as*

$$f_{\mathrm{H}}(\alpha) = \dim(K_\alpha) \qquad \textit{and} \qquad f_{\mathrm{P}}(\alpha) = \mathrm{Dim}(K_\alpha),$$

*respectively. We also introduce their continuous versions:*

$$f_{\mathrm{H,c}}(\alpha) := \lim_{\varepsilon\to 0} \dim(K_{\alpha-\varepsilon,\alpha+\varepsilon}) \qquad \textit{and} \qquad f_{\mathrm{P,c}}(\alpha) := \lim_{\varepsilon\to 0} \mathrm{Dim}(K_{\alpha-\varepsilon,\alpha+\varepsilon}).$$

The continuous versions are, by definition, more regular than the usual ones. $f_{\mathrm{H,c}}$ has been studied by Lau & Ngai [LN] in the context of infinite Bernoulli convolutions and a closely related notion has appeared earlier in a work by Brown, Michon and Peyrière [BMP, Thm. 2].

Of practical interest is yet another approach to multifractal analysis. Based on a partition of $\mathbb{R}^d$, we will define two *coarse multifractal spectra* $f_{\mathrm{G}}$ and $f_{\mathrm{L}}$. For simplicity we stick to the case $d = 1$; the general case is obvious.

**Definition 4** *Let $H_\delta$ be the set of all intervals $B = [l\delta, (l+1)\delta)$ with integer $l$ and with $\mu(B) \neq 0$. Let $B_1 := [(l-1)\delta, (l+2)\delta)$. The* grid spectrum *is defined as*

$$f_{\mathrm{G}}(\alpha) := \lim_{\varepsilon\to 0} \limsup_{\delta\to 0} \frac{\log N_\delta(\alpha,\varepsilon)}{\log 1/\delta}$$

*where*

$$N_\delta(\alpha,\varepsilon) = \#\{B \in H_\delta \,:\, |B_1|^{\alpha+\varepsilon} \leq \mu(B_1) < |B_1|^{\alpha-\varepsilon}\}.$$

Here, $N_\delta$ denotes the number of 'intervals from a grid of size $\delta$ with coarse Hölder exponent $\alpha(B) = \log\mu(B)/\log|B|$ roughly equal to $\alpha$'. As was described earlier in [R1], the straightforward or naive way of counting intervals gives poor results in theory as well as in numerical application. Among the various possible improvements suggested by Strichartz, Olsen, Lau & Ngai, Arbeiter & Patzschke, and one of the present authers [S, LN, O, AP], we favor the given one for its simplicity and accuracy [R1, PR].

Though tempting it is *wrong* to interpret $f_{\mathrm{G}}$ as the box dimension of $K_\alpha$ (Ex. 1). The truth is that $K_\alpha$ has the same box dimension as its topological closure which is, in the case of self-similar measures, equal to the whole support of the measure. In fact, recalling $K_{\alpha,\alpha'} = G_\alpha \cap F_{\alpha'}$ and setting

$$A_m := \{t \in [0,1] \,:\, |I|^{\alpha+2\varepsilon} \leq \mu(I) < |I|^{\alpha-2\varepsilon} \text{ if } t \in I \text{ and } |I| \leq 1/m\} \tag{5}$$

yields

$$N_\delta(\alpha, 2\varepsilon) \geq \#\{B \in H_\delta \,:\, B \cap A_m \neq \emptyset\}, \tag{6}$$

provided $3\delta < 1/m$. Denoting the box dimension of a bounded set $A$ by $\Delta(A)$ we have

$$\Delta(A_m) := \limsup_{\delta\to 0} \frac{\log \#\{B \in H_\delta \,:\, B \cap A_m \neq \emptyset\}}{\log 1/\delta} \leq \limsup_{\delta\to 0} \frac{\log N_\delta(\alpha, 2\varepsilon)}{\log 1/\delta}.$$

It is well known that $\mathrm{Dim}(\cdot) \leq \Delta(\cdot)$ (see Tricot or Falconer [Tr, F]). Together with $\cup_m A_m \supset K_{\alpha-\varepsilon,\alpha+\varepsilon}$ and $\mathrm{Dim}(\cup_m A_m) = \sup_m \mathrm{Dim}(A_m)$, one concludes $f_{\mathrm{G}}(\alpha) \geq f_{\mathrm{P,c}}(\alpha)$. In combination with $\dim(\cdot) \leq \mathrm{Dim}(\cdot)$ [F, Tr], we obtain the following relation between the various spectra:

**Lemma 5** $f_{\mathrm{G}}(\alpha) \geq f_{\mathrm{P,c}}(\alpha) \geq f_{\mathrm{P}}(\alpha) \geq f_{\mathrm{H}}(\alpha)$ *and* $f_{\mathrm{G}}(\alpha) \geq f_{\mathrm{P,c}}(\alpha) \geq f_{\mathrm{H,c}}(\alpha) \geq f_{\mathrm{H}}(\alpha)$.

If the box dimension was $\sigma$-stable like Hausdorff and packing dimension, one could argue $f_{\mathrm{G}}(\alpha) \geq \sup_m \Delta(A_m) = \Delta(\cup_m A_m) = \Delta(\mathrm{supp}(\mu))$ which is obviously not true. The spectrum $f_{\mathrm{G}}$ is related to the *partition function* $\tau(q)$

$$\tau(q) := \liminf_{\delta\to 0} \frac{\log \sum_{B\in H_\delta} \mu(B_1)^q}{\log \delta}$$

through the Legendre transform [R1]

$$\tau(q) = \inf_{\alpha\in\mathbb{R}} \Big(q\alpha - f_{\mathrm{G}}(\alpha)\Big). \tag{7}$$

This relation holds also in the much more general context of Choquet capacities (see Levy-Vehel and Vojak [LV, Thm 3]). The tentative inversion formula (2) translates to:

$$q^\dagger = -\tau \qquad\qquad \tau^\dagger = -q. \tag{8}$$

Most evidently it holds for self-similar measures (compare (4)). In general, however, (8) will fail as is the case with discontinuous self-similar measures. It may also fail for continuous measures, e.g. if their spectrum $f_{\mathrm{G}}$ is not strictly concave.

**Definition 6** *It is natural to introduce the Legendre transform of $\tau(q)$ as a multifractal spectrum:*

$$f_{\mathrm{L}}(\alpha) := \inf_{q\in\mathbb{R}} \Big(q\alpha - \tau(q)\Big).$$

An equivalent form of (7) is to say that $f_{\mathrm{L}}$ is the concave hull of $f_{\mathrm{G}}$. Consequently:

**Lemma 7** $f_{\mathrm{G}}(\alpha) \leq f_{\mathrm{L}}(\alpha)$.

For typical values of $\alpha$, we have equality. In fact [R2]:

$$\begin{aligned} f_{\mathrm{G}}(\alpha^+) &= q\alpha^+ - \tau(q) \qquad && (q>0) \\ f_{\mathrm{G}}(\alpha^-) &= q\alpha^- - \tau(q) \qquad && (q<0) \end{aligned} \tag{9}$$

where $\alpha^+ := \tau'(q+)$ and $\alpha^- := \tau'(q-)$ denote the one-sided derivatives of $\tau(q)$.

The multifractal formalism is closely related to the thermodynamical formalism and means that equality holds in Lemma 5. To establish it under various assumptions has been a point of major interest in multifractal analysis (see Kahane & Peyrire, Collet et al, Rand, Pesin & Weiss and aforementioned authors [KP, CLP, Ra, LN, AP, O, R1, PW]). In general, however, the estimate (6) can clearly be sharp, meaning that an interval $B$ can show a coarse Hölder exponent $\alpha(B) = \alpha$ although it contains no point $t$ with $\alpha(t) = \alpha$. The most simple example is the absolutely continuous measure $\mu$ with density $\phi(t) = t$ on $[0,1]$, i.e. $M(t) = t^2/2$. Here, $\alpha(t) = 1$ for $0 < t \leq 1$ and $\alpha(0) = 2$, hence $f_{\mathrm{H}}(1) = 1$, $f_{\mathrm{H}}(2) = 0$ and $K_\alpha$ is empty otherwise. A direct calculation shows, on the other hand, that $f_{\mathrm{G}}(\alpha) = 2 - \alpha$ for $1 \leq \alpha \leq 2$. What seems to be a paradox is readily explained: while $\log \mu(I)/\log |I|$ tends to 1 for all $t > 0$, a coarse graining on any 'pre-asymptotic' level $\delta > 0$ will show a non–trivial distribution of Hölder exponents. The inequality $f_{\mathrm{G}} > f_{\mathrm{H}}$ is a direct consequence of the highly non-uniform convergence of the Hölder exponents $\alpha(t)$.

Further examples of a similar kind are present with the inverse measures of self-similar measures. Before introducing them in Section 3, we provide some intuition on inverse measures by giving the proof of the inversion formula (2) in the continuous case.

## 2.2 The continuous case

By saying loosely that we are in the 'continuous case' we mean that

$$M(t) = \mu([0,t]) \text{ is continuous and strictly increasing.} \tag{10}$$

Equivalently, we could require that one of the following conditions are satisfied:
i) $\mu$ and $\mu^\dagger$ are both continuous.
ii) $M : [0,1] \mapsto [0,1]$ is onto and one-to-one with inverse $M^\dagger$.

Provided (10) holds and $0 < \alpha < \infty$, we have

$$t \in K_\alpha \Leftrightarrow M(t) \in K^\dagger_{1/\alpha}, \tag{11}$$

or, more generally,

$$M(F_\alpha) = G^\dagger_{1/\alpha} \qquad\qquad M(G_\alpha) = F^\dagger_{1/\alpha}.$$

This is a simple consequence of $|M(I)| = \mu(I)$ and $\mu^\dagger(M(I)) = |I|$ which holds for arbitrary intervals $I$ due to (10).

**Proposition 8** *Let $\mu$ be a probability measure on $[0,1]$ and let $A$ be a subset of $G_\alpha$ $(0 < \alpha < \infty)$. Then,*

$$\dim(A) \geq \alpha \cdot \dim(M(A)).$$

**Proof**

Fix $\alpha' < \alpha$ and let $A_m = \left\{x \in A : \mu(I) \leq |I|^{\alpha'} \text{ if } x \in I \text{ and } |I| \leq 1/m\right\}$. Obviously, $A = \cup_{m \geq 1} A_m$ .

Let $a$ denote the left boundary point of an interval $I$. Then, $|M(I)| = \mu(I \backslash \{a\}) \leq \mu(I)$ since $M$ is right continuous. Let $\{I_j\}_j$ be a covering of $A_m$ by intervals of length less than $1/n$ $(n > m)$. For the $I_j$ which intersect $A_m$, we have

$$|M(I_j)| \leq \mu(I_j) \leq |I_j|^{\alpha'} \leq (1/n)^{\alpha'}.$$

Consequently, $\{M(I_j) : I_j \cap A_m \neq \emptyset\}$ forms a covering of $M(A_m)$ by intervals of length less than $\delta_n := (1/n)^{\alpha'}$ and we find

$$\eta_{\delta_n}^{\gamma/\alpha'}\Big(M(A_m)\Big) \leq \sum_{I_j \cap A_m \neq \emptyset} |M(I_j)|^{\gamma/\alpha'} \leq \sum_j |I_j|^\gamma.$$

Taking the infimum over all coverings of $A_m$ we find

$$\eta_{\delta_n}^{\gamma/\alpha'}\Big(M(A_m)\Big) \leq \eta_{1/n}^{\gamma}(A_m) \leq \eta^\gamma(A_m) \leq \eta^\gamma(A),$$

thus, $\dim(M(A_m)) \leq \dim(A)/\alpha'$. With the $\sigma$-stability of Hausdorff dimension, i.e. $\dim(M(A)) = \sup_m \dim(M(A_m))$, the claim follows by letting $\alpha' \nearrow \alpha$. $\diamondsuit$

**Proposition 9** *Assume that $M$ is continuous and strictly increasing. Then*

$$\mathrm{Dim}(A) \leq \alpha \cdot \mathrm{Dim}(M(A))$$

*for any subset $A$ of $F_\alpha$, provided* $0 \leq \alpha < \infty$.

**Proof**

In its basic structure, this proof is very similar to the one of Proposition 8 above. Note that $\alpha = 0$ is allowed. Fix $\alpha' > \alpha$ and let

$$A_m = \left\{x \in A : \mu(I) \geq |I|^{\alpha'} \text{ if } x \in I \text{ and } |I| \leq 1/m\right\}.$$

Consider a $1/n$-packing $\{I_j\}_j$ of $A_m$ which is a collection of mutually disjoint, open intervals, each of length less or equal $1/n$ and each intersecting $A_m$. Since $M$ and $M^\dagger = M^{-1}$ are continuous, the collection of all $M(I_j)$ provides a packing of $M(A_m)$. The central estimate is

$$|M(I_j)| = \mu(I_j) \geq |I_j|^{\alpha'}$$

which holds due to (10). All that is needed to get the obvious argumentation started is an upper estimate of the length of $M(I_j)$. Once more we use the continuity of $M$, more precisely its uniform continuity. Choose $\delta > 0$. Then there is $n$ such that $|I| \leq 1/n$ implies $|M(I)| \leq \delta$.

In summary, $\{M(I_j)\}_j$ is a $\delta$-packing of $M(A_m)$ provided $n$ is sufficiently large. This allows to estimate the $\gamma$-dimensional packing premeasure $\hat{\pi}$:

$$\hat{\pi}_\delta^\gamma\Big(M(A_m)\Big) \geq \sum |M(I_j)|^\gamma \geq \sum |I_j|^{\gamma\alpha'}.$$

It is an easy task now to complete the proof in a similar way as above. $\diamondsuit$

**Corollary 10 (Inversion formula in the continuous case)** *Assume that $M$ is onto and one-to-one. Let $0 < \alpha < \infty$, and let $A$ be any subset of $K_\alpha$. Then,*

$$\dim(A) = \alpha \cdot \dim(M(A)), \qquad \textit{and} \qquad \mathrm{Dim}(A) = \alpha \cdot \mathrm{Dim}(M(A)).$$

*Finally, $K_0$ is at the most of dimension $0$ but might be empty:*

$$\dim(K_0) = \mathrm{Dim}(K_0) \leq 0.$$

Consequently,

$$f_{\mathrm{H}}^{\dagger}(\alpha) = \dim(K_\alpha^{\dagger}) = \dim(M(K_{1/\alpha})) = \alpha \dim(K_{1/\alpha}) = \alpha f_{\mathrm{H}}^{\dagger}(1/\alpha)$$

and similar for $f_{\mathrm{P}}$.

**Remark 11** In the continuous case, $f_{\mathrm{G}}$ is properly linked with the spectrum $f_F$ obtained by the so-called 'fixed mass algorithm', provided $\tau(q)$ is a strictly concave diffeomorphism [R2]. As its name suggests, $f_F$ is obtained through a partition of $[0,1]$ into intervals of equal mass. This partition translates immediately into a usual grid on the $\theta$-axis. As a consequence, the inversion formula holds in this case also for $f_{\mathrm{G}}$ and $f_{\mathrm{L}}$.

**Proof**
Note first that $M(A) \subset K_{1/\alpha}^{\dagger}$ and that $M^{\dagger}(M(A)) = A$ due to (11). Applying Proposition 8 once to $\mu$ and $A \subset K_\alpha \subset G_\alpha$, and once to $\mu^{\dagger}$ and $M(A) \subset K_{1/\alpha}^{\dagger} \subset G_{1/\alpha}^{\dagger}$ yields $\dim(A) \geq \alpha \dim(M(A)) \geq \dim(M^{\dagger}(M(A))) = \dim(A)$. The argument for $\mathrm{Dim}(\cdot)$ is similar.

$\diamondsuit$

**Remark 12** Proposition 8 could be used to establish the inversion formula in general if it were not for a generalization of (11) which appears to be cumbersome. In the context of Section 4, this generalization will come more natural.

**Remark 13** In the definition of $K_\alpha, F_\alpha, \ldots$, all possible intervals are considered. In certain situations, however, it is convenient to restrict the attention to a family $\mathcal{J}$ of intervals. Then, if $K_\alpha, F_\alpha$ and $G_\alpha$ are defined using only elements of $\mathcal{J}$ , the sets $K_{\alpha^\dagger}^{\dagger}$, $F_{\alpha^\dagger}^{\dagger}$, and $G_{\alpha^\dagger}^{\dagger}$ have to be defined using the family $M(\mathcal{J})$ of intervals on the $\theta$-axis. The definition of dimension has then to be modified accordingly on the $t$- and the $\theta$-axes.

## 3 Discontinuous self-similar maesures

In this section, we provide the full multifractal analysis of a broader class of self-similar measures, allowing also discontinuous ones. As a corollary, we obtain the inversion formula (2) for $f_{\mathrm{H}}$ and $f_{\mathrm{P}}$ in this special case as well as a counter example showing that (2) may fail for $f_{\mathrm{G}}$ and $f_{\mathrm{L}}$. Moreover, we obtain a weak form of the multifractal formalism for the discontinuous self-similar measures, namely, that the 'coarse' spectrum $f_{\mathrm{G}}$ is the concave hull of the 'fine' spectrum $f_{\mathrm{H}}$.

What might look like a loss can be turned into a gain: Coarse multifractal analysis of the inverse measure of a given measure may provide the information hidden in the linear part of $f_{\mathrm{G}}$—as is the case with discontinuous self-similar measures. (Note that this procedure is *not* equivalent to the fixed mass algorithm which is as sensitive to the presence of atoms as $f_{\mathrm{G}}$ [R2].)

We conclude the section by comparing discontinuous self-similar measures with equilibrium measures of dynamical systems.

## 3.1 Extended notion of self-similar measures

We start with two simple examples.

**Example 1 [Failure of the Multifractal Formalism]** Consider the self-similar measure $\mu_C$ invariant under the maps $w_0(t) = r_0 t$ and $w_1(t) = (1-r_1) + r_1 t$ with $r_0 + r_1 < 1$ and with probabilities $p_0 = p_1 = 1/2$ (see (3)). By definition, intervals of zero $\mu_C$ measure correspond to atoms of the inverse measure ${\mu_C}^\dagger$. Since their lengths add up to 1, ${\mu_C}^\dagger$ must be purely atomic. A closer look reveals that ${\mu_C}^\dagger$ consists of a hierarchy of atoms situated in the binary points 1/2, 1/4, 3/4, 1/8, etc. having masses $r_2 := 1 - r_0 - r_1$, $r_0 r_2$, $r_1 r_2$, $r_0 r_0 r_2$, $r_0 r_1 r_2$, $r_1 r_0 r_2$, $r_1 r_1 r_2$, etc.

Introducing a third map $w_2(t) = r_0 + r_2 t$ with probalility $p_2 = 0$ leaves $\mu_C$ unchanged. The inverse measure ${\mu_C}^\dagger$, on the other hand, is then invariant under $w_0^\dagger(\theta) = \theta/2$, $w_2^\dagger(\theta) = p_0 + p_2 \cdot \theta \equiv 1/2$ and $w_1^\dagger(\theta) = p_0 + p_2 + p_1 \cdot \theta = 1/2 + \theta/2$ with probabilities $r_0$, $r_2$ and $r_1$, respectively.

Though purely atomic, ${\mu_C}^\dagger$ possesses non-trivial spectra since its support is not countable. By Corollary 22, (8) and (4), the fine multifractal spectra $f_{\mathrm{H}}^\dagger$ and $f_{\mathrm{P}}^\dagger$ of ${\mu_C}^\dagger$ are composed of the origin and a bell-shaped curve which is the graph of the Legendre transform of

$$\beta^\dagger(s) = -\log_2 ({r_0}^s + {r_1}^s).$$

This curve has maximal value 1 and touches the line of slope $D$ through the origin, $D$ being the dimension of the support of $\mu_C$, i.e. the zero of $\beta^\dagger$.

The rough estimate

$$\sum_{B \in H_{1/2^n}} \mu(B_1)^s \simeq {r_2}^s \sum_{k=0}^{n-1} \sum_{\varepsilon_1 \ldots \varepsilon_k \in \{0,1\}^k} (r_{\varepsilon_1} \cdot \ldots \cdot r_{\varepsilon_k})^s = {r_2}^s \frac{1 - ({r_0}^s + {r_1}^s)^n}{1 - {r_0}^s - {r_1}^s},$$

which is made precise in Proposition 18, implies

$$\tau^\dagger(s) = \begin{cases} \beta^\dagger(s) & \text{for } s \leq D \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the inversion formula (8) holds exactly for $q \geq 0$, i.e. for $\tau \geq -D$. By Theorem 19

$$f_{\mathrm{G}}^\dagger(\alpha) = f_{\mathrm{L}}^\dagger(\alpha) = \begin{cases} D \cdot \alpha & \text{for } 0 \leq \alpha \leq (\beta^\dagger)'(D), \\ f_{\mathrm{H}}^\dagger(\alpha) & \text{for } \alpha = (\beta^\dagger)'(s) \text{ and } s < D. \end{cases}$$

Note that $f_{\mathrm{G}}^\dagger(0) = 0$ by direct calculation. This is in stark contrast to the fact that the set of atoms is dense, hence, of box dimension 1. ○

**Example 2 [Failure of inversion formula for the coarse graining approach]** Take $p_0 \in (0,1)$ and let $p_1 = 1 - p_0$. Consider the multifractal measure $\mu$ composed of Dirac measures $p_0^n \cdot p_1$ in the points $1/2^n$:

$$\mu = \sum_{n=0}^{\infty} p_1 p_0^n \cdot \delta_{\{2^{-n}\}}.$$

Note first that $\mu$ is invariant under $w_0(t) = t/2$ and $w_1(t) \equiv 1$ (compare (3)). As before, it is convenient to add a map $w_2(t) = t/2 + 1/2$ with probability $p_2 = 0$ to the invariance family of $\mu$.

For the fine multifractal spectra $f_{\rm H}$ and $f_{\rm P}$, we find: $f(0) = 0$, $f(\infty) = 1$ and $f(\alpha_0) = 0$ for $\alpha_0 = -\log p_0/\log 2$, which is the Hölder exponent at $t = 0$. For all other $\alpha$, we have $K_\alpha = \emptyset$.

Straightforward calculation yields $\tau(q) = \alpha_0 q$ for $q < 0$ and $\tau(q) = 0$ otherwise. Also by explicit calculation or using Theorem 19, one finds $f_{\rm G}(\alpha) = 0$ for $0 \le \alpha \le \alpha_0$ and $f_{\rm G}(\alpha) = -\infty$ otherwise.

By drawing a graph of $M$, it is easy to see that $\mu^\dagger$ is of the same form as $\mu$:

$$\mu^\dagger = \sum_{n=1}^{\infty} 2^{-n} \cdot \delta_{\{p_0^n\}}.$$

In conclusion, the inversion formula (2) can been verified for the fine multifractal spectra $f_{\rm H}$ and $f_{\rm P}$, but if fails for the coarse multifractal spectra $f_{\rm G}$ and $f_{\rm L}$ in all points but $\alpha = \alpha_0$. For $\tau(q)$, the inversion formula (8) holds only at $q = 0$.

○

Consider the following definition of a self-similar measure $\mu$ on $[0,1]$ which is broader than the usual one [H, CM, R1]:

**Definition 14** *A probability measure* $\mu$ *on* $[0,1]$ *is called* self-similar *iff*

$$\mu(E) = \sum_{i=0}^{u-1} p_i \mu(w_i^{-1}(E)), \tag{12}$$

*where* $w_i$ *is a similarity map of* $[0,1]$ *into itself with* contraction ratios $r_i$, *and where we require that* $r_0+\ldots+r_{u-1} = 1$, $p_0+\ldots+p_{u-1} = 1$, $r_i \ge 0$ *and* $p_i \ge 0$ *for all* $i$. *Furthermore, we call* $\mu$ discontinuous self-similar *iff* $r_i = 0 \ne p_i$ *and* $r_j \ne 0 \ne p_j$ *for some* $i$ *and some* $j$.

If $p_i = 0$ for all $i$ with $r_i = 0$, then $\mu$ is self-similar in the usual sense [H]. Allowing $p_i = 0$ means to include measures supported on a set of dimension strictly less than 1. Allowing $r_i = 0$ means to include the inverse of such self-similar measures. The condition $r_i = 0 \ne p_i$ implies that there are atoms while $r_j \ne 0 \ne p_j$ avoids the triviality of $\mu$ reducing to a finite number of atoms. Discontinuous self-similar measures are purely atomic: by $n$-fold application of (12) the mass not lying in an atom is smaller than $(\sum_{r_i \ne 0} p_i)^n$ which tends to zero.

Here, we stretch the notion of self-similarity beyond its original meaning that 'the whole' can be 'regained' by enlarging any little part of it. Still, these measures are invariant, unique, and they can be obtained by 'redistributing mass' in intervals in a self-similar way. In particular, the cylindrical sets $V_{\sigma|n} = w_{\sigma_1} \ldots w_{\sigma_n}((0,1))$ obtained by iteratively applying the maps $w_i$ continue to be useful when approximating the measure $\mu$: $\mu(V_{\sigma|n}) = \xi_n p_{\sigma_1} \cdot \ldots \cdot p_{\sigma_n}$ with $c \le \xi_n \le 1/c$ for some constant $c > 0$.

It is not necessary to use maps to produce the sets $V_{\sigma|n}$ and one can think of a more general construction of measures through a nested family of sets $V_{\sigma|n}$, sometimes called Moran constructions. As is shown in [CM, R1, PW], the multifractal spectra do not depend on the actual positioning of 'daughter sets' $V_{\sigma|n+1}$ within $V_{\sigma|n}$ as long as the obvious separation condition is respected. Applying the inversion formula in its general form (Theorem 21), we conclude that the actual masses of the atoms ($p_i > 0 = r_i$) of a discontinuous measure are not essential but the 'multiplicative process' which rules the length and mass of the intervals that seperate them. The spectra will, therefore, depend only on the non-degenerate entries, i.e. the maps with $r_i \ne 0 \ne p_i$.

We need adopt the separation condition.

**Definition 15** *Given a self-similar measure, the* open set condition *is said to hold with $K$ iff $K$ is compact with nonempty interior $O$ such that $w_i(O)$ are mutually disjoint subsets of $K$.*

For ordinary self-similar measures, this definition coincides with the usual one, e.g. with the one used in [AP]. The fine multifractal spectra of a discontinuous self-similar measure (12) can be computed in the straightforward generalization of (4):

**Theorem 16** *Let $\mu$ be a self-similar measure and define the concave, differentiable function $\beta$ through*

$$\sum_{i:r_i\neq 0\neq p_i} {p_i}^q {r_i}^{-\beta(q)} = 1. \tag{13}$$

*Assume that the OSC is satisfied with $K=[0,1]$. Then,*

$$f_{\rm H}(\alpha) = f_{\rm H,c}(\alpha) = f_{\rm P}(\alpha) = f_{\rm P,c}(\alpha) = \beta^*(\alpha) := q\beta'(q) - \beta(q)$$

*at $\alpha = \beta'(q)$ for $q \in \mathbb{R}$ as well as for $q \to \pm\infty$. For all other $\alpha \in (0,\infty)$, we have $K_\alpha = \emptyset$. $K_0$ is at most countable and it is non-empty iff there is $i$ with $r_i = 0 \neq p_i$. $K_\infty$ contains nonempty open intervals iff it is non-empty and iff there is $j$ with $p_j = 0 \neq r_j$.*

**Remark 17** The theorem holds also in the random case in the sense of [AP], given that assumption 1.1 iii) of [AP] is replaced by: iii') there is a number $r_{\rm min} > 0$ such that $r_i$ is either 0 or larger than $r_{\rm min}$ with probability 1 and similar for $p_i$.

In our context, infinite Hölder exponents occur only in gaps. We include them for reasons of symmetry and completeness. In general, infinite Hölder exponents may occur also as non–trivial limits. As an example, we refer to the left sided multifractal presented in [MEH, RM1]. Some of these infinite self-similar measures are continuous and non–vanishing, and have Hölder exponent $\infty$ (Lebesgue) almost everywhere [RM1, Ex. 1].

**Proof**

Using the inequalities between the various spectra as stated in Lemma 5, it is enough to show that $f_{\rm P,c} \leq \beta^*$ and $\beta^* \leq f_{\rm H}$.

We think of the points $t \in [0,1]$ as being encoded by a sequence $\sigma = \sigma_1\sigma_2\ldots$ in the usual way, i.e. $\sigma \in \Sigma := \{0,\ldots,u-1\}^{\mathbb{N}}$ and the sequence $w_{\sigma|n}(0) := w_{\sigma_1} \circ \ldots \circ w_{\sigma_n}(0)$ converges to $t$. The coding is unique for all but countably many points $t$ (if $r_i = 1/10$ for all $i$, then this is just the usual decimal representation). We denote by $\Sigma^r := \{\sigma \in \Sigma : r_{\sigma_i} \neq 0 \neq p_{\sigma_i}\}^{\mathbb{N}}$. All but a countable number of points $t$ on supp$(\mu)$, e.g. the atoms of $\mu$, are encoded with sequences of $\Sigma^r$. Note, that sequences from $\Sigma^r$ can also encode atoms.

Some notation is useful: $\Sigma_n := \{0,\ldots,u-1\}^n$, $\Sigma_* := \bigcup_n \Sigma_n$, and similarly $\Sigma^r_n$ and $\Sigma^r_*$. Let $\sigma|n := \sigma_1\ldots\sigma_n$. It will be clear from the context whether $\sigma|n$ is an arbitrary word of length $n$ or whether it is the beginning segment of length $n$ of a given longer word. Let $r_{\sigma|n} = r_{\sigma_1}\cdot\ldots\cdot r_{\sigma_n}$, $p_{\sigma|n} = p_{\sigma_1}\cdot\ldots\cdot p_{\sigma_n}$, and

$$J_\delta := \{\sigma|n \in \Sigma_* \,:\, r_{\sigma|n} \leq \delta < r_{\sigma|n-1} \text{ and } p_{\sigma|n} \neq 0\} \qquad J^r_\delta := J_\delta \cap \Sigma^r_*. \tag{14}$$

These sets $J_\delta$ can be thought of as being constructed iteratively in the following way. Arrange the set of non-vanishing $r_{\sigma|n}$ in non-increasing order and let $\delta_m$ be the $m$-th value in this ordering. For convenience, set $\delta_0 := 1$ and $r_\emptyset := 1$. Then, induction starts with $J_{\delta_0} = \emptyset$, $J_{\delta_1} = \{0,\ldots,u-1\}$, and $J_{\delta_{k+1}}$ is obtained from $J_{\delta_k}$ by replacing words $\sigma|n$

with $r_{\sigma|n} = \delta_k$ by all extensions $\sigma_1 \dots \sigma_{n+1}$ with $p_{\sigma_{n+1}} \neq 0$ ($r_{\sigma_{n+1}} = 0$ is allowed). Finally, $J_\delta = J_{\delta_m}$ for $m$ such that $\delta_m \leq \delta < \delta_{m-1}$.

Let $V_{\sigma|n} := w_{\sigma|n}((0,1))$. By induction, it is easy to see [H, R1] that $\{V_{\sigma|n} : \sigma|n \in J^r_\delta\}$ forms a cover of all points with address in $\Sigma^r$ and that

$$\sum_{\sigma|n \in J^r_\delta} {p_{\sigma|n}}^q {r_{\sigma|n}}^{-\beta(q)} = 1 \tag{15}$$

for every $J^r_\delta$. Moreover, $|V_{\sigma|n}| = r_{\sigma|n}$ and $\mu(V_{\sigma|n})/\mu((0,1)) = p_{\sigma|n}$.

Now, it is easy to check that the claim is true for Hölder exponents 0 and $\infty$: First, $\alpha(t)$ is bounded from above for all $t$ provided $p_i > 0$ for all $i$. This follows easely by considering the intervals $I_n$ defined to be the $\underline{r}^n$ parallelbody of $V_{\sigma|n}$ for any sequence $\sigma$ that encodes $t$, and by noting $I \ni t$, $|I_n| \leq 3\overline{r}^n$, and $\mu(I) \geq \mu((0,1)) \cdot p_{\sigma|n}$ (here, $\underline{r}$ and $\overline{r}$ denote the smallest, nonvanishing and the largest $r_i$ respectively). Thus, if $K_\infty \neq \emptyset$ there exists necessarily a $j$ with $p_j = 0 \neq r_j$. But if so, the interior of $K_\infty$ is obviously not empty.

For Hölder exponent 0 note that $\alpha(t)$ is bounded from below by $\min_i(\log p_i / \log r_i)$ provided $r_i > 0$ for all $i$.

Assume for the remainder that $0 < \alpha < \infty$. Let $\varepsilon > 0$ such that $\alpha - \varepsilon > 0$. The coding sequence of a point of $K_{\alpha-\varepsilon,\alpha+\varepsilon}$ must belong to $\Sigma^r$ by definition. For this restricted set of digits, the usual arguments apply as we are about to show.

For the upper bound one considers $A_m$ as defined in (5). For $m$ large enough and $\delta < 1/m$, a cover of $A_m$ is formed by $\{V_{\sigma|n} : \sigma|n \in J^r_\delta \text{ and } {p_{\sigma|n}}^q \geq {r_{\sigma|n}}^{(q\alpha+3|q\varepsilon|)}\}$. Then,

$$1 \geq \sum_{\sigma|n \in J^r_\delta} {p_{\sigma|n}}^q {r_{\sigma|n}}^{-\beta(q)} \geq \sum_{\sigma|n \in J^r_\delta} {r_{\sigma|n}}^{q\alpha+3|q\varepsilon|-\beta(q)} = \sum_{\sigma|n \in J^r_\delta} (|V_{\sigma|n}|)^{q\alpha+3|q\varepsilon|-\beta(q)}.$$

implies $\mathrm{Dim}(A_m) \leq \Delta(A_m) \leq q\alpha + 3|q\varepsilon| - \beta(q)$. Taking the infimum over all $q$ we obtain $\mathrm{Dim}(A_m) \leq \max(\beta^*(\alpha+3\varepsilon), \beta^*(\alpha-3\varepsilon))$. With

$$\mathrm{Dim}(K_{\alpha-\varepsilon,\alpha+\varepsilon}) \leq \mathrm{Dim}(\cup_m A_m) = \sup_m \mathrm{Dim}(A_m)$$

the upper bound $f_{\mathrm{P,c}} \leq \beta^*$ follows by letting $\varepsilon \to 0$. The random case can be treated as in [AP].

To obtain the lower bound, consider the invariant measure $\Phi_q$ defined by (12) where the $p_i$ have been replaced by $\overline{p}_i := {p_i}^q {r_i}^{-\beta(q)}$. Here, only letters from $\Sigma^r_1$ need to be considered since $0 < \alpha < \infty$. We would like to apply the results of [AP]. (The random case reduces trivially to the determistic case when choosing Dirac distributions for the random variables.) Since we don't have $\sum_{r_i \neq 0} p_i = 1$, the various steps in [AP] need to be verified. First, we do have $\Phi_q(\partial K) = 0$ for all $q \in \mathbb{R}$. Next, the strong open set condition holds for the family $w_i$ ($i \in \Sigma^r_1$) since it has been shown to be equivalent to the OSC by Schief [Sch] and by Patzschke [Pa]. At this point, we may conclude already that $\Phi_q$ concentrates on the points $t$ with address $\sigma \in \Sigma^r$ such that

$$\frac{\log p_{\sigma|n}}{\log r_{\sigma|n}} \to \alpha_q = \beta'(q) \quad \text{and} \quad \frac{\log \overline{p}_{\sigma|n}}{\log r_{\sigma|n}} \to q\alpha_q - \beta(q) = \beta^*(\alpha_q).$$

Now, we claim that $\alpha(t)$ can be computed using $V_{\sigma|n}$, $\Phi_q$ a.s. Then, it follows that $K_{\alpha_q}$ itself has full $\Phi_q$ measure, thus, positive $\beta^*(\alpha_q)$-dimensional Hausdorff measure. We proceed as in [AP, Lemma 3.8]. Some caution is needed, though, since mass may lie on the boundary points of $K_{\sigma|n} = w_{\sigma|n}([0,1])$. Rather than with $K_{\sigma|n}$, we have to argue with $V_{\sigma|n} = w_{\sigma|n}((0,1))$. Due to the OSC $\mu(V_{\sigma|n}) = p_{\sigma|n} \cdot \mu((0,1))$. Since $V_{\sigma|n}$ is the interior of $K_{\sigma|n}$, we may substitute the basic estimates $B(h(\sigma), r) \supset K_{\sigma|k_r(\sigma)}$ and $B(h(\sigma), r) \subset K_{\sigma|\tilde{k}_r(\sigma)}$ in [AP] by $B(h(\sigma), r) \supset V_{\sigma|k_r(\sigma)}$ and $B(h(\sigma), r/2) \subset V_{\sigma|\tilde{k}_r(\sigma)}$. This is obviously sufficient for the estimation of Hölder exponents. (Hereby, $h(\sigma)$ denotes the point $t$ with address $\sigma$). Together with $|V_{\sigma|n}| = r_{\sigma|n}$ the claim follows as in [AP]. It relies heavily on the fact that the distance of a point to the boundary of $K$ is log-integrable with respect to $\Phi_q$. In other words, points of $K_{\alpha_q}$ do not come too often too close to the atoms of $\mu$.

$\diamondsuit$

In order to compute the coarse multifractal spectra, let us first investigate $\tau(q)$.

**Proposition 18** *Let $\mu$ be a discontinuous self-similar measure. Define $\beta$ as in (13) and denote its zero by $D^\dagger$. Assume that the OSC is satisfied with $K = [0,1]$. Then, the partition function $\tau(q)$ of $\mu$ satisfies*

$$\tau(q) = \begin{cases} \beta(q) & q \leq D^\dagger \\ 0 & \textit{otherwise.} \end{cases}$$

**Proof**

To avoid trivialities, we discard with letters $i$ such that $r_i = p_i = 0$. We use the notation of the proof of Theorem 16.

Due to its self-similarity, (12) $\mu$ possesses atoms: denoting by $a_i$ the fixed point of $w_i$ we have $\mu(\{a_i\}) \geq p_i$ if $r_i = 0$. By Definition 14, there is at least one atom, i.e. $p_i > 0$. As a matter of fact, $\mu$ consists entirely of atoms. We won't use this fact, though.

The exact values $m_i := \mu(\{a_i\}) > 0$ ($a_i \in A_0$) are not important and depend on the fact whether 0 and/or 1 are atoms. The OSC implies disjointness of the sets $V_{\sigma|n} := w_{\sigma|n}((0,1))$ for $\sigma|n \in \Sigma_n^r$. But overlap may occur for other $\sigma|n$, i.e. for atoms: The OSC can not be iterated for the sets $V_i$ which are contained in the boundary of $K$. The partition function $\tau(q)$, describing a scaling behavior, depends not on $m_i$, but only on the way how the further atoms are produced by the multiplicative process as one iterates (12) in order to obtain more detailed information about $\mu$.

Assume first that $\mu(\{0\}) = \mu(\{1\}) = 0$. See Ex. 1. Consider the set $J_\delta$ as defined in (14) and recall its iterative construction. The following remarks are most easily established by induction. First, the set

$$J_\delta^a := J_\delta \backslash J_\delta^r = \{\sigma|n \in J_\delta \,:\, r_{\sigma_n} = 0\}$$

encodes atoms. More precisely, the sets $V_{\sigma|n}$ with $\sigma|n \in J_\delta^a$ are singletons and the tails $\sigma_{n+1}, \sigma_{n+2}, \ldots$ are of no significance since $r_{\sigma_n} = 0$. The set $J_\delta^r$, on the other hand, encodes mutually disjoint open intervals $V_{\sigma|n}$ of positive length $r_{\sigma|n}$ . Between any two atoms of $J_\delta^a$ lies an open interval $V_{\sigma|n}$ ($\sigma|n \in J_\delta^r$). The sets $V_{\sigma|n}$ ($\sigma|n \in J_\delta$) cover the support of $\mu$ up to finite many points of zero measure. We have

$$\mu(V_{\sigma|n}) = \begin{cases} p_{\sigma|n} \cdot \mu((0,1)) & \text{if } \sigma|n \in J_\delta^r, \\ p_{\sigma|n-1} m_{\sigma_n} & \text{if } \sigma|n \in J_\delta^a. \end{cases} \tag{16}$$

Finally, still by induction, the mass of an arbitrary atom $a = w_{\sigma|n}(0)$ ($\sigma|n \in J_\delta^a$) is comparable to the mass of an entire neighborhood of $a$:

$$\mu(\{a\}) = \mu(V_{\sigma|n}) \leq \mu([a - \underline{r}\delta, a + \underline{r}\delta]) \leq c_0 \mu(\{a\}). \tag{17}$$

where $\underline{r} = \min\{r_i : r_i \neq 0\}$ and $c_0 = 1 + 2\mu((0,1))/\min_i(m_i))$ are constants. To see this, assume for a moment that $J_\delta$ had been constructed allowing also words with $p_{\sigma|n} = 0$. Then, the sets $V_{\sigma'|m}$ $(\sigma'|m \in J_\delta)$ cover all of $[0,1]$ up to finite many points of zero measure. By induction, $a$ is a boundary point to two open intervals $V_{\sigma'|m}$ and $V_{\sigma''|k}$ $(\sigma'|m, \sigma''|k \in J_\delta^r)$ with $m,k \geq n$ and $\sigma'|n-1 = \sigma''|n-1 = \sigma|n-1$. This implies $\mu(V_{\sigma'|m})/\mu((0,1)) = p_{\sigma'|m} \leq p_{\sigma|n-1} = \mu(\{a\})/m_{\sigma_n}$. Since $V_{\sigma'|m}$ and $V_{\sigma''|k}$ are of length at least $\underline{r}\delta$, the claim follows easily.

In the general case, that is if we allow atoms in 0 and/or 1, the list of atoms $a_i$ at 'first stage' (boundary points of the intervals $V_i$ $(i \in \Sigma_1^r)$ with positive mass) will contain not only the fixpoints of maps $w_i$ with $r_i = 0 < p_i$. We may still have $m_i := \mu(\{a_i\}) = p_i$ (see Ex. 2) and the arguments above are valid. In general, however, overlap will occur on the boundary of $V_i$ $(i \in \Sigma_1)$ leading to $m_i > p_i$ for some of the atoms $a_i$ at first stage. If so, we have to adopt the definition of $J_\delta$ slightly: in the iterative construction of $J_\delta$, a 'newly arriving' atom $V_{\sigma|n}$ may coincide with an already existing one, say $V_{\sigma'|m}$, which must lie on the boundary of the parent $V_{\sigma|n-1}$. Consequently, $\sigma_n$ must encode one of the atoms in 0 and/or 1 and $m < n$, $\sigma'|m-1 = \sigma|m-1$. In this case we keep only the shorter address $\sigma'|m$ and discard $\sigma|n$ (the additional mass supposed to arrive at $V_{\sigma|n}$ was already accounted for by $m_{\sigma'_m}$).

It is important to note that we may assume without loss of generality that there are atoms of the form $a_i = w_i(a_i) \in (0,1)$ at 'first stage': if not, we use that $\mu$ is also invariant under the family $w_{ij}$. The claim follows then from the very definition of discontinuous self-similar measures (Definition 14) and by choosing an enumeration for $\Sigma_2$. This said, we hurry to add that (16) and (17) hold in general.

In order to compute $\tau(q)$, it is convenient to estimate $S_\delta(q) := \sum_{B \in H_\delta} \mu(B_1)^q$ against $\sum_{J_{\delta'}} p_{\sigma|n}{}^q$ for some $\delta'$ which is a multiple of $\delta$.

Consider an interval $B \in H_\delta$. Choose $\delta' = \delta$. Since $\mu(B) \neq 0$ by definition, we find a set $V_{\sigma|n}$ $(\sigma|n \in J_\delta)$ intersecting $B$ and conclude $\mu(B_1) \geq \mu(V_{\sigma|n}) \geq c_1 p_{\sigma|n}$, where $c_1 > 0$ is a constant. Thus, $S_\delta(q) \leq c_1{}^q \sum_{J_{\delta'}} p_{\sigma|n}{}^q$ for $q < 0$.

In addition, $B_1$ intersects at the most $c_2 := 1 + 3/\underline{r}$ sets $V_{\sigma|n}$ with $\sigma|n \in J_{\delta'}^r$. Consequently, $B_1$ contains at the most the same number of atoms $V_{\sigma|n}$ with $\sigma|n \in J_{\delta'}^a$. Let $\sigma|n(B)$ denote the word corresponding to the maximum of these masses, i.e. $\sigma|n(B) := \operatorname{argmax}\{\mu(V_{\sigma|n}) : \sigma|n \in J_{\delta'}$ and $B_1 \cap V_{\sigma|n} \neq \emptyset\}$. Then, $\mu(B) \leq \mu(B_1) \leq 2c_2 p_{\sigma|n(B)}$. Since every $w_{\sigma|n}((0,1))$ $(\sigma|n \in J_\delta)$ can intersect at the most 4 intervals $B_1$, we conclude $S_\delta(q) \leq 4(2c_2)^q \sum_{J_{\delta'}} p_{\sigma|n}{}^q$ for $q \geq 0$.

Now, consider $\sigma|n \in J_{\delta'}$. Assume first that $\sigma|n \in J_{\delta'}^a$ and choose $\delta = \delta'\underline{r}^2/3$. Since $V_{\sigma|n}$ is a singleton formed by an atom, there is $B \in H_\delta$ which contains it. By (17)

$$1/c_3 p_{\sigma|n} \leq \mu(V_{\sigma|n}) \leq \mu(B_1) \leq c_0\mu(V_{\sigma|n}) \leq c_3 p_{\sigma|n} \tag{18}$$

for some constant $c_3$. If, on the other hand, $\sigma|n \in J_{\delta'}^r$ pick an atom $V_{\sigma|n+1}$ in $V_{\sigma|n}$. A set of this form exists since there is an atom $a_i = w_i(a_i)$ in $(0,1)$. There is $B \in H_\delta$ which contains $V_{\sigma|n+1}$. By choice of $\delta$, $B_1 \subset V_{\sigma|n}$ and $m_{\sigma_{n+1}} p_{\sigma|n} \leq \mu(B_1) \leq c_0 m_{\sigma_{n+1}} p_{\sigma|n}$. Thus, increasing $c_3$ if necessary the same estimate (18) holds.

The argument just given is, of course, closely related to the fact that $\mu$ is an infinite sum of atoms. In summary

$$\tau(q) = \liminf_{\delta \to 0} \frac{\log \sum_{J_\delta} p_{\sigma|n}{}^q}{\log \delta}.$$

For the asymptotical behavior of this sum, note first that the words $\sigma|n \in J_\delta^r$ are of length $n$ between $\log\delta/\log\underline{r}$ and $\log\delta/\log\overline{r}$ with $\overline{r} = \max\{r_i\}$. For every word $\sigma|n \in J_\delta^a$, on the other hand, there is $\sigma'|m \in J_\delta^r$ with $\sigma|n-1 = \sigma'|n-1$.

Assume first that $q > D^\dagger$. Then, there is $i$ such that

$$p_i^q \le \sum_{J_\delta} p_{\sigma|n}{}^q \le \sum_{n=0}^{\infty}\sum_{\Sigma_n^r} p_{\sigma|n}{}^q = \sum_{n=0}^{\infty}\Big(\sum_{i\in\Sigma_1^r} p_i{}^q\Big)^n < \infty,$$

thus, $\tau(q) = 0$. For $q = D^\dagger$, in a similar way $p_i^q \le \sum_{J_\delta} p_{\sigma|n}{}^q \le \log\delta/\log\overline{r}$ and $\tau(D^\dagger) = 0$.

If $q < D^\dagger$, then

$$\delta^{-\beta(q)}\sum_{J_\delta} p_{\sigma|n}{}^q \ge \delta^{-\beta(q)}\sum_{J_\delta^r} p_{\sigma|n}{}^q = \xi(\delta)\sum_{J_\delta^r} p_{\sigma|n}{}^q r_{\sigma|n}{}^{-\beta(q)} = \xi(\delta) \tag{19}$$

where we used (15) and where $\xi(\delta)$ is bounded between $\min\{1, \underline{r}^{\beta(q)}\}$ and $\max\{\underline{r}^{\beta(q)}, 1\}$ for all $\delta$. Thus, $\tau(q) \le \beta(q)$.

Finally, we estimate the sum over $J_\delta^a$ from above in a very crude way. Including a factor $\xi = \sum_i m_i^q$, we can discard with the last digits of such words $\sigma|n$ and replace them with $\sigma|n-1$. Then, $r_{\sigma|n-1} > \delta$, and since $\beta(q) < 0$

$$\begin{aligned}\delta^{-\beta(q)}\sum_{\sigma|n\in J_\delta^a} p_{\sigma|n}{}^q &\le \xi \sum_{\sigma|n\in J_\delta^a} p_{\sigma|n-1}{}^q r_{\sigma|n-1}{}^{-\beta(q)} \le \xi \sum_{n=0}^{\log\delta/\log\overline{r}}\ \sum_{\sigma|n\in\Sigma_n^r} p_{\sigma|n}{}^q r_{\sigma|n}{}^{-\beta(q)} \\ &\le \xi' \log\delta/\log\overline{r}.\end{aligned}$$

Together with (15), we obtain $\tau(q) \ge \beta(q)$. This completes the proof. In fact it was shown that $\tau(q)$ assumes the limit $\delta \to 0$. ♢

**Theorem 19** *The grid spectrum $f_{\rm G}$ of a self-similar measure (12) equals the Legendre transform $f_{\rm L}(\alpha)$ of $\tau(q)$.*

The formula for $\tau(q)$ made already clear that the multifractal formalism must break down for discontinuous self-similar measures in one or the other way: $f_{\rm L}$ contains a linear part of slope $D^\dagger$. The graph of $f_{\rm H}$ and $f_{\rm P}$, on the other hand, consists of the origin and and a strictly concave curve which touches the line of slope $D^\dagger$ through the origin (Theorem 16). Due to Theorem 19, the damage is even worse: also $f_{\rm G}$, which contains in general more information than the partition function $\tau(q)$ [R1], does not provide the full singularity spectrum $f_{\rm H}$.

**Corollary 20** *The multifractal formalism does not apply to discontinuous self-similar measures, i.e. $f_{\rm H} = f_{\rm P} \ne f_{\rm G} = f_{\rm L}$. A weaker form holds, though: ${f_{\rm H}}^{**} = f_{\rm L}$.*

This comes to its extreme with measures the fine multifractal spectra of which consist of only two points: the grid spectrum is a line connecting these two points (see Ex. 2 and a degenerate case of Ex. 1).

**Proof**

The 'classical' case is well known [R1] and we may assume that $\mu$ is a discontinuous self-similar measure. The upper bound $f_{\rm G} \le f_{\rm L}$ holds in general. For $\alpha = 0$, this implies immediately $f_{\rm G}(0) = 0$ which can as well be obtained by direct computation. It remains to provide a lower bound on $f_{\rm G}(\alpha)$ for $\alpha > 0$. For notation, we refer to Proposition 18.

Let $h \in (0,1)$ and set $\delta' = \delta^h$ and $\delta'' = \delta \underline{r}^2/3$.

Consider a word $\sigma|n \in J^r_{\delta'}$. For $\delta > 0$ small enough, $r_{\sigma|n} \geq \underline{r}\delta' > \delta$ and $J^a_\delta$ contains all $\sigma|n+1$ with $r_{\sigma_{n+1}} = 0$. For each such atom, there is $B \in H_{\delta''}$ with (18). For each $\sigma|n \in J^r_{\delta'}$, select one such interval and denote it by $B(\sigma|n)$. Since $B(\sigma|n) \subset V_{\sigma|n}$, this is unique within $J^r_{\delta'}$.

Assuming now $r^{\alpha+\varepsilon}_{\sigma|n} \leq p_{\sigma|n} \leq r^{\alpha-\varepsilon}_{\sigma|n}$ we find $\mu(B_1) \leq c_3 \delta'^{\alpha-\varepsilon} \leq c_4 |B_1|^{h(\alpha-\varepsilon)} \leq |B_1|^{h(\alpha-2\varepsilon)}$, provided $\delta < 1/r^2 \cdot c_4{}^{-1/2h\varepsilon}$. Here we use that $c_4$ depends on $\alpha$, $h$, and $\varepsilon$ but not on $\delta$. Similarly, $\mu(B_1) \geq |B_1|^{h(\alpha+2\varepsilon)}$, provided $\delta$ is small enough.

Let $J^r_{\delta'}(\alpha,\varepsilon) := \{\sigma|n \in J^r_{\delta'} : r^{\alpha+\varepsilon}_{\sigma|n} \leq p_{\sigma|n} \leq r^{\alpha-\varepsilon}_{\sigma|n}\}$. As we will show in a moment, a large deviation result allows to conclude that

$$\lim_{\varepsilon\to 0} \lim_{\delta'\to 0} \frac{\log \# J^r_{\delta'}(\alpha,\varepsilon)}{\log 1/\delta'} = \beta^*(\alpha)$$

In fact, the proof is formally identical with the one given in [R1, Thm 7]. Since

$$\frac{\log N_\delta(h\alpha, 2h\varepsilon)}{\log 1/\delta} \geq \frac{\log \# J^r_{\delta'}(\alpha,\varepsilon)}{\log 1/\delta} = h \frac{\log \# J^r_{\delta'}(\alpha,\varepsilon)}{\log 1/\delta'}$$

we conclude that $f_{\mathrm{G}}(\alpha) \geq h\beta^*(\alpha/h)$. This proves the theorem.

In order to apply a large deviation result of Ellis-Gärtner [E], we recall the asymptotic behavior of the partition function corresponding to $J^r_\delta$. By (19): $\sum_{J^r_\delta} p_{\sigma|n}{}^t = \xi(\delta)\delta^{-\beta(t)}$ where $\xi(\delta)$ is bounded.

Now, consider the probability spaces $J^r_{\delta_n}$ with uniform distribution where $\delta_n \to 0$ is an arbitrary sequence. Denote the moment generating function of the random variables $X_n = \log p_{\sigma|n}$ by $c_n(t) := \mathbb{E}[\exp(tX_n)] = \sum_{J^r_{\delta_n}} p_{\sigma|n}{}^t / \# J^r_{\delta_n}$. Let $a_n := \log \delta_n \to -\infty$. Then, we have

$$c(t) := \lim_{n\to\infty} \frac{1}{a_n} \log c_n(t) = \beta(t) - \beta(0).$$

Since $c$ is concave and differentiable, [E, theorem II.2] applies: denote by $P_n(U)$ the probability that $(1/a_n)X_n$ lies in $U$ for a randomly picked $\sigma|n$. If $U$ is open and $U'$ is closed, then

$$I(U) \leq \liminf_{n\to\infty} \frac{\log P_n(U)}{-a_n} \qquad \limsup_{n\to\infty} \frac{\log P_n(U')}{-a_n} \leq I(U')$$

where $I(U) := \sup\{I(\alpha) : \alpha \in U\}$ and $I(\alpha) = \inf_t (t\alpha - c(t)) = \beta^*(\alpha) + \beta(0)$.

Choosing $U = (\alpha - \varepsilon/2, \alpha + \varepsilon/2)$ and $U' = [\alpha - 2\varepsilon, \alpha + 2\varepsilon]$ we have $P_n(U) \cdot \# J^r_{\delta_n} \leq \# J^r_{\delta_n}(\alpha,\varepsilon) \leq P_n(U') \cdot \# J^r_{\delta_n}$ for $n$ large enough. $\Diamond$

## 3.2 Equilibrium measures

A natural generalization of the notion of self-similar measures are the equilibrium measures which appear in the theory of dynamical systems. In a typical situation on the line, one will consider a conformal mapping $g$ which maps some disjoint intervals $I_i \subset [0,1]$ onto $[0,1]$ such that $-\log|g'|$ is negative and Hölder continuous. The invariant measure $\mu$ in question will then live on the repeller of $g$, more precisely it will be the equilibrium measure of another Hölder continuous function $\phi$. This scheme reduces to the self-similar case if $g$ is such that the $w_i$ are its inverse branches and if $\phi$ takes the constant value $\log p_i$ on $I_i$.

The multifractal formalism, which basically states that $f_{\mathrm{H}}(\alpha) = f_{\mathrm{L}}(\alpha)$, has been established for Cookie-cutters by Rand [Ra], and for equilibrium measures of certain Moran constructions by Pesin and Weiss [PW]. Set $\psi = \exp(\phi - P\{\phi\})$ with $P$ denoting the pressure function and let $\beta$ be (uniquely) defined through $P\{q \log \psi - \beta(-\log|g'|)\} = 0$. Then, $\tau$ equals $\beta$, and the spectra of $\mu$ collapse with the Legendre transform $\beta^*$. Note, that the definition of $\beta$ reduces to the usual one (13) in the self-similar case.

It is tempting to produce new measures analogously to self-similar measures, i.e. to exchange the roles of 'geometry' $-\log|g'|$ and 'mass' $\phi$, and to compare this procedure with the inversion. Assume, therefore, that $\phi = -\log|h'|$ for some function $h$ with properties analogous to $g$. Denote the $h$-invariant equilibrium measure corresponding to $\overline{\phi} := -\log|g'|$ by $\overline{\mu}$.

The fine multifractal spectra of $\mu^\dagger$ can be obtained through the inversion formula, i.e. they equal the Legendre transform of the inverse $\beta^{-1}$. In analogy with Proposition 18, we conjecture that the partition function of $\mu^\dagger$ is $\min\{\beta^{-1}, 0\}$.

Being an equilibrium measure, $\overline{\mu}$ has its fine multifractal spectra equal to $\overline{\beta}^*$ where, as before, $P\{t \log \overline{\psi} - \overline{\beta}(-\log|h'|)) = 0$ with $\overline{\psi} = \exp(\overline{\phi} - P\{\overline{\phi}\})$. Though very closely related, the spectra of $\mu^\dagger$ and $\overline{\mu}$ are very well distinguished, i.e. $\overline{\beta} \neq \beta^{-1}$, unless $P\{\phi\}$ and $P\{\overline{\phi}\}$ vanish. But this is the degenerate case when $\overline{\mu}$ and $\mu$ are supported on all of $[0,1]$.

One particular difference between the spectra of $\mu^\dagger$ and $\overline{\mu}$ is the slope of their tangent through the origin, i.e. the zero of $\beta^{-1}$ and $\overline{\beta}$, respectively. With the continuous $\overline{\mu}$, this slope is 1 while it is strictly less than 1 for the discontinuous $\mu^\dagger$. This fact reflects the fundamentally different way of dealing with the fact of 'loosing mass' when approximating the measure iteratively by $\mu(V_{\sigma|n})$. With $\mu^\dagger$, loss of mass in the generating process is compensated by producing atoms. To the contrary with $\overline{\mu}$ which is 'renormalized' in each step by a factor $e^{-P}$ in order to prevent it from dying out or exploding (compare [Ra, p 389]). (For the equilibrium measure $\overline{\mu}$, the sets $V_{\sigma|n}$ are obtained iteratively as the components of the sets $h^{-1}(V_{\sigma|n-1})$.) This re-normalization brings a shift in the Hölder exponents which causes the distinct yet closely related shape of the spectra of $\mu^\dagger$ and $\overline{\mu}$.

It is this different way of compensating mass which causes the failure of the multifractal formalism for the inverse measure $\mu^\dagger$.

# 4 The inversion formula in the general case

This section is devoted to the general proof of the inversion formula for $f_{\mathrm{H}}$ and $f_{\mathrm{P}}$. For notation, we refer back to Section 2. Our main result is

**Theorem 21** *Let $\mu$ be a probability measure on $[0,1]$ and $\mu^\dagger$ its inverse measure. Assume $0 < \alpha \leq \alpha' < \infty$. Then,*

$$1/\alpha \cdot \dim(K_{\alpha,\alpha'}) \geq \dim(K^\dagger_{1/\alpha',1/\alpha}) \geq 1/\alpha' \cdot \dim(K_{\alpha,\alpha'})$$

*and*

$$1/\alpha' \cdot \mathrm{Dim}(K_{\alpha,\alpha'}) \leq \mathrm{Dim}(K^\dagger_{1/\alpha',1/\alpha}) \leq 1/\alpha \cdot \mathrm{Dim}(K_{\alpha,\alpha'}).$$

**Corollary 22 (Inversion formula)** *Let $\mu$ be a probability measure on $[0,1]$ and $\mu^\dagger$ its inverse measure. Assume that $0 < \alpha < \infty$. Then,*

$$f^\dagger(\alpha) = \alpha f(1/\alpha)$$

*where $f$ may stand for $f_{\mathrm{H}}$, $f_{\mathrm{P}}$, $f_{\mathrm{H,c}}$, or $f_{\mathrm{P,c}}$.*

**Proof**

**o) The plan** It is possible to apply the arguments given for $f_{\mathrm{H}}$ in the continuous case to general measures (see Proposition 8). Difficulties arise, however, if some of the atoms of $\mu$ lie on the boundary of gaps, the main problems lying in a generalization of $M(K_\alpha) = K^\dagger_{1/\alpha}$. In addition, the argument for $f_{\mathrm{P}}(\alpha)$ cannot be generalized in this way because there is no one-to-one correspondence between packings of $K_\alpha$ and $K^\dagger_{1/\alpha}$ in the presence of gaps. It is worthwhile, therefore, to give the following, somewhat more elaborate argument which proves the inversion formula in full generality for the Hausdorff spectrum and the packing spectrum.

The first step i) consists in perturbing $\mu$ slightly to obtain a new measure $\mu^p$ which is non–vanishing. The corresponding $M^p(t) = \mu^p([0,t])$ is strictly increasing but not necessarily continuous.

As will be shown in ii)-iii), $\mu^p$ and $\mu$ have the same Hölder exponents in all points of interest. More precisely, we have $K^p_{\alpha,\alpha'} \cap \mathcal{R} = K_{\alpha,\alpha'} \cap \mathcal{R}$, where

$$\mathcal{R} := \{t \in [0,1] : \mu(I_n) \to 0 \Leftrightarrow |I_n| \to 0 \text{ for all sequences } (I_n) \text{ with } t \in I_n \ \forall n\}.$$

We call the points of $\mathcal{R}$ *$\mu$-regular*. Restricting attention to $\mathcal{R}$ means, in particular, to exclude the points in the gaps of $\mu$ which would contribute the $\mu^p$-Hölder exponent 1. Non–regular points either belong to the closure of some gap or are an atom of $\mu$. Therefore, $K_{\alpha,\alpha'} \backslash \mathcal{R}$ is at most countable and the spectra $f_{\mathrm{H}}$ and $f_{\mathrm{P}}$ of $\mu$ are not affected by replacing $K_{\alpha,\alpha'}$ by $K_{\alpha,\alpha'} \cap \mathcal{R}$. For $\mu^p$, on the other hand, excluding points outside $\mathcal{R}$ changes the spectrum. Here, we will take advantage of the fact that the inversion formula holds for subsets of $K^p_{\alpha,\alpha'}$.

The change from $\mu^\dagger$ to $\mu^{p\dagger} := (\mu^p)^\dagger$ corresponds to an expansion $\Psi$ on the $\theta$-axis which we introduce in iv). It is, unfortunately, not globally bi-Lipschitz. On each $G^\dagger_\alpha$, however, the distortion is small enough to preserve dimension. This is shown in v)-vii).

Once it is established that the perturbation does not affect the spectra, we simply apply the same procedure to $\nu := \mu^{p\dagger}$. This produces $\nu^p$ which is continuous and non-vanishing by construction. The inversion formula holds, thus, for $\nu^p$ which has the same dimension spectra as $\nu = \mu^{p\dagger}$ and, hence, the same as $\mu^\dagger$. Its inverse $\nu^{p\dagger}$ has the same spectra as $\nu^\dagger = \mu^{p\dagger\dagger} = \mu^p$, which coincide with the spectra of $\mu$. Through this chain of equalities,carried out in detail in viii), we will obtain the desired result.

**i) The perturbed measure $\boldsymbol{\mu^p}$** Let $\phi(\varepsilon) := \varepsilon^{1/\varepsilon}$. Let $\mathcal{A}$ denote the countable, possibly empty set of values which $M$ takes more than once. For notational simplicity, we reserve the letter $a$ for elements of $\mathcal{A}$. For every $a$ let $L_a := \{t : M(t) = a\}$, a so-called *gap*, which is an interval closed to the left and open or closed to the right. Let

$$\mu^p := \mu + \sum_a \lambda_a$$

where $\lambda_a$ is an absolutely continuous measure on $L_a$ defined as follows: if the boundary points of $L_a$ are denoted by $s < t$ then $\lambda_a([s, s+h]) = \lambda_a([t-h, t]) = \phi(h)$ for $0 \leq h \leq |L_a|/2$. The total mass added to $\mu$ in $L_a$ is $m_a := \lambda_a(L_a) = 2\phi(|L_a|/2) \leq \phi(|L_a|)$. Outside of the gaps, $M^p(t) := \mu^p([0,t])$ increases strictly since $M$ does, inside a gap it is differentiable with derivative $\phi'(h) > 0$, $h$ being the distance to the boundary of the gap.

**ii) Comparing $\boldsymbol{\mu}$ and $\boldsymbol{\mu^p}$** Let $I$ be an interval of length $l \leq 1$ which is not contained in any gap, in other words, which contains a point from $\mathcal{R}$. Let $l_a := |L_a \cap I|$ for all

$a \in \mathcal{A}$. Due to $l_a \le l$, we have $\sum_{l_a>0} \phi(l_a) \le \sum l_a^{1/l} = l^{1/l} \sum (l_a/l)^{1/l} \le l^{1/l} \sum (l_a/l) \le l^{1/l}$ from which we conclude

$$\mu(I) \le \mu^p(I) \le \mu(I) + \phi(|I|). \tag{20}$$

For all sufficiently small intervals $I$ containing a point $t$ of $K_{\alpha,\alpha'}$, $\mu(I)$ will eventually be larger than $|I|^{2\alpha'}$ and hence larger than $\phi(|I|)$. Relying on this idea, we will prove the claims announced in o). From (20), it follows also that $\mu^p$ has total mass $\mu^p(\mathbb{R}) \in [1,2]$. We refrain from normalizing $\mu^p$ for the sake of simplicity.

**iii) Hölder exponents of $\boldsymbol{\mu^p}$** Consider a sequence of intervals $I_n$ which converges down to $t \in \mathcal{R}$. Assume that

$$\alpha(I_n) := \frac{\log \mu(I_n)}{\log |I_n|} \to \alpha$$

and take $\varepsilon \in (0,\alpha)$. If $|I_n|$ is sufficiently small, we have $\phi(|I_n|) \le |I_n|^{\alpha+\varepsilon} \le \mu(I_n)$. With (20),

$$\mu^p(I_n) \le 2\mu(I_n), \tag{21}$$

implying $\alpha^p(I_n) := \log \mu^p(I_n)/\log |I_n| \to \alpha$. Assume, on the other hand, that $\alpha^p(I_n) \to \alpha$. For sufficiently small $|I_n|$, we find $\phi(|I_n|) \le |I_n|^{\alpha+2\varepsilon} \le (1/2)|I_n|^{\alpha+\varepsilon} \le (1/2)\mu^p(I_n)$ and conclude with (20) that $\alpha(I_n) \to \alpha$. Altogether,

$$\alpha(I_n) \to \alpha \quad \text{if and only if} \quad \alpha^p(I_n) \to \alpha, \tag{22}$$

from which

$$K^p_{\alpha,\alpha'} \cap \mathcal{R} = K_{\alpha,\alpha'} \cap \mathcal{R},$$

provided $0 < \alpha \le \alpha' < \infty$. Note that we need $t \in \mathcal{R}$ in order to obtain (21).

**iv) Inverse measure $\boldsymbol{\mu^{p\dagger}}$ of $\boldsymbol{\mu^p}$** In order to compare $\mu^\dagger$ and $\mu^{p\dagger}$, which can be regarded as a 'perturbation' of the former, we introduce the expanding map $\Psi$ on the $\theta$-axis which identifies the points $M(t)$ and $M^p(t)$. On $\mathcal{R}^\dagger := M(\mathcal{R})$, we may define $\Psi = M^p \circ M^{-1}$, or, more generally

$$\Psi : \mathbb{R}\backslash\mathcal{A} \to \mathbb{R} \qquad \theta \mapsto \theta^p := \theta + \sum_{a<\theta} m_a$$

with $m_a = \lambda_a(L_a)$. To avoid confusion, we will use the superscript $^p$ for objects in the image-space of $\Psi$.

$\Psi$ is continuous on $\mathbb{R}\backslash\mathcal{A}$ because $\sum_{|a-\theta|\le 1/n} m_a \to 0$ for all $\theta \notin \mathcal{A}$. Obviously, there is no continuous extension to $\mathcal{A}$, the atoms of $\mu^\dagger$. Each $a \in \mathcal{A}$ is 'stretched' into a whole interval

$$L_a^p := [u_a, v_a] := [\sup_{\theta<a} \Psi(\theta), \inf_{\theta'>a} \Psi(\theta')]$$

which is of length $m_a = \lambda_a(L_a)$. On the boundary of $L_a^p$, the Hölder exponent of $\mu^{p\dagger}$ is infinite, in the interior it is 1. It is clear that we have to exclude these points from our considerations. Since

$$\mathcal{A} \subset \mathbb{R}\backslash\mathcal{R}^\dagger = \mathbb{R}\backslash M(\mathcal{R}) \qquad \text{and} \qquad L_a^p = M^p(\overline{L_a}) \subset \mathbb{R}\backslash M^p(\mathcal{R}),$$

this will happen automatically, so to say, by restricting our attention to $\mathcal{R}$ and $\mathcal{R}^\dagger$, the $\mu$ and the $\mu^\dagger$-regular points. It is useful to denote the set of these points in the $^p$-space (which are the ones of interest to us) by

$$\mathcal{R}^{p\dagger} := \Psi(\mathcal{R}^\dagger) = \Psi(M(\mathcal{R})) = M^p(\mathcal{R}). \tag{23}$$

Here, we are slightly inconsequent in our notation since $\mathcal{R}^{p\dagger}$ is not the entire set of $\mu^{p\dagger}$-regular points but only the ones that do not lie in any $L^p_a$.

Note some simple properties. Let $I^p$ be an interval and let $I$ be the convex hull of its pre-image under $\Psi$, i.e.

$$I := \langle\Psi^{-1}(I^p)\rangle := \Psi^{-1}(I^p) \cup \{a \in \mathcal{A} \,:\, L^p_a \subset I^p\} \tag{24}$$

which is again an interval. Denote by $\hat{I}$ the interior of $I$ and by $\overline{I}$ its closure. The definitions imply

$$|\overline{I}| = |\hat{I}| \le |I^p| = |\hat{I}| + \sum_{a\in\overline{I}} |L^p_a \cap I^p| \le |\hat{I}| + \sum_{a\in\overline{I}} m_a \tag{25}$$

and

$$\mu^\dagger(\hat{I}) \le \mu^{p\dagger}(I^p) \le \mu^\dagger(\overline{I}) \tag{26}$$

with equality in (26) unless $I^p$ ends in some $L^p_a$, in other words, unless an atom lies on the boundary of $I$. The essential ingredient for the remainder is a translation of the 'error estimate' (21) used in iii). To this, we note

$$\sum_{a\in I} m_a \le \sum_{a\in I} \phi(|L_a|) \le \phi(\sum_{a\in I} |L_a|) \le \phi(\mu^\dagger(I)) \tag{27}$$

for all intervals $I$, which follows using first the same argument as in (20) and finally $|L_a| = \mu^\dagger(\{a\})$.

**v) Comparing $K^\dagger_{\alpha,\alpha'}$ and $K^{p\dagger}_{\alpha,\alpha'}$** ($K^{p\dagger}_{\alpha,\alpha'} := G^{p\dagger}_\alpha \cap F^{p\dagger}_\alpha$ is the obvious set for $\mu^{p\dagger}$.) The basic idea is clear: The term $\sum m_a$ in (25) can be neglected due to (27) as soon as an upper estimate of $\mu^\dagger(I)$ against $|I|$ or $|I^p|$ is available. If so, Hölder exponents must be identical. Minor difficulties arise, however, from the fact that some details of intervals $I^p$ on the $\theta^p$-axis are not reflected by $\langle\Psi^{-1}(I^p)\rangle$, in particular when $I^p$ ends in some $L^p_a$.

Take $\theta \in G^\dagger_\alpha \cap \mathcal{R}^\dagger$, $\varepsilon \in (0, \alpha)$ and let $\theta^p := \Psi(\theta)$. Take an interval $I^p \ni \theta^p$ of length smaller than $1/n$ and let $I := \langle\Psi^{-1}(I^p)\rangle$. Certainly, $\theta \in I$ and $|\overline{I}| \le |I^p| \le 1/n$. Assume that $n$ has been chosen large enough to ensure $\mu^\dagger(\overline{I}) \le |\overline{I}|^{\alpha-\varepsilon}$ and $\phi((1/n)^{\alpha-\varepsilon}) \le 1/n$. The latter implies $\phi(x) \le x^{1/(\alpha-\varepsilon)}$ whenever $0 \le x \le (1/n)^{\alpha-\varepsilon}$. With $x = \mu^\dagger(\overline{I})$ and (27), we obtain $\sum_{a\in\overline{I}} m_a \le \mu^\dagger(\overline{I})^{1/(\alpha-\varepsilon)} \le |\overline{I}|$. Thus, (25) and (26) combine to

$$\frac{\log\mu^\dagger(\overline{I})}{\log|\overline{I}|} \le \frac{\log\mu^{p\dagger}(I^p)}{\log|I^p|} \le \frac{\log\mu^\dagger(\hat{I})}{\log 2|\hat{I}|} \tag{28}$$

which proves $\theta^p \in G^{p\dagger}_\alpha$. Moreover, (28) provides the desired knowledge on the accumulation points of $\alpha(I^p)$. With (23), we get $\Psi(K^\dagger_{\alpha,\alpha'} \cap \mathcal{R}) \subset K^{p\dagger}_{\alpha,\alpha'} \cap \mathcal{R}^{p\dagger}$.

For convenience, we repeat the assumptions for (28): $\theta \in G^\dagger_\alpha \cap \mathcal{R}^\dagger$, $I^p \ni \theta^p := \Psi(\theta)$, and $I^p$ of sufficiently small length.

**vi)** Let $\theta^p \in G^{p\dagger}_\alpha \cap \mathcal{R}^{p\dagger}$ and take $\varepsilon \in (0, \alpha/2)$. The argument we will give is almost identical to the one in v) only that we estimate $\sum_a m_a$ against $\mu^{p\dagger}(I^p)$. For later use in vii), we start again with $I^p \ni \theta^p$ and let $I := \langle\Psi^{-1}(I^p)\rangle$. Unlike (28), we have to produce an estimate involving $\mu^\dagger(I)$ rather than $\mu^\dagger(\overline{I})$ or $\mu^\dagger(\hat{I})$. So, we have to deal with the possibility of $I^p$ having a boundary point in some $L^p_a$.

Assume that $|I^p| \le 1/n$ where $n$ is large enough to ensure

$$\mu^{p\dagger}(I^p) \le |I^p|^{\alpha-\varepsilon}, \qquad \phi((1/n)^{\alpha-2\varepsilon}) \le 1/n, \qquad \text{and} \qquad n \ge 2^{\varepsilon/(\alpha-2\varepsilon)}.$$

We have $\phi(x) \le x^{1/(\alpha-2\varepsilon)}$ for all $x \le (1/n)^{\alpha-2\varepsilon}$ and $|I^p|^{(\alpha-\varepsilon)/(\alpha-2\varepsilon)} \le (1/2)|I^p|$. Consider an arbitrary $L_a^p$ which intersects $I^p$. By assumption $\theta^p \notin L_a^p = [u_a, v_a]$, in other words, $I^p$ must contain a boundary point of $L_a^p$. Say we have $L_a^p \cap I^p = [u_a, w_a]$ for some $w_a \le v_a$. (The argument is similar in the symmetric case.) By construction, $L_a^p = M^p(\overline{L_a})$ and there are $s < t$ with $M^p(s) = u_a$, $M^p(t) = w_a$. Since $M^p$ is one-to-one, $s$ must be the left boundary point of $L_a$ from which

$$w_a - u_a = |L_a^p \cap I^p| = \mu^p((s,t)) \le \phi(t-s) = \phi(\mu^{p\dagger}((u_a, w_a)))$$

and

$$\sum_{a\in\overline{I}} |L_a^p \cap I^p| \le \sum_{a\in\overline{I}} \phi(\mu^{p\dagger}((u_a, w_a))) \le \phi\Big(\sum_{a\in\overline{I}} \mu^{p\dagger}((u_a, w_a))\Big) \le \phi(\mu^{p\dagger}(I^p))$$

follow. Choose $x = \mu^{p\dagger}(I^p)$. Then, $x \le |I^p|^{\alpha-\varepsilon} \le (1/n)^{\alpha-2\varepsilon}$ and

$$\phi(\mu^{p\dagger}(I^p)) \le \mu^{p\dagger}(I^p)^{1/(\alpha-2\varepsilon)} \le 1/2|I^p|.$$

With (25)

$$|I| \le |I^p| \le 2|I|. \tag{29}$$

For convenience, we repeat the assumptions of (29): $\theta^p \in G_\alpha^{p\dagger} \cap \mathcal{R}^{p\dagger}$, $I^p \ni \theta^p$, and $I^p$ of sufficiently small length.

This bound is all we will need in vii) to estimate dimensions. To conclude on Hölder exponents, however, we have to estimate $\mu^\dagger(I)$ against $\mu^{p\dagger}(I^p)$ which is not possible under such general assumptions. Fortunately, we need only consider the following situation: take any interval $I$ containing $\theta := \Psi^{-1}(\theta^p)$ and let

$$I^\Psi := \langle \Psi(I) \rangle := \Psi(I) \cup \bigcup_{a\in I} L_a^p. \tag{30}$$

Then, $\mu^\dagger(I) = \mu^{p\dagger}(I^\Psi)$ by (26), and $I = \langle \Psi^{-1}(I^\Psi) \rangle$. Substituting $I^\Psi$ for $I^p$ in (29) we obtain, for $|I^\Psi|$ small,

$$\frac{\log \mu^{p\dagger}(I^\Psi)}{\log(|I^\Psi|/2)} \le \frac{\log \mu^\dagger(I)}{\log |I|} \le \frac{\log \mu^{p\dagger}(I^\Psi)}{\log |I^\Psi|} \le \frac{\log \mu^\dagger(I)}{\log(2|I|)}. \tag{31}$$

But, letting $I \searrow \theta$ implies $I^\Psi \searrow \theta^p$ since $\sum_{a\in I} m_a \to 0$ ($\Psi$ is continuous). Together with v) we conclude:

$$G_\alpha^{p\dagger} \cap \mathcal{R}^{p\dagger} = \Psi\Big(G_\alpha^\dagger \cap \mathcal{R}^\dagger\Big) \qquad \text{and} \qquad K_{\alpha,\alpha'}^{p\dagger} \cap \mathcal{R}^{p\dagger} = \Psi\Big(K_{\alpha,\alpha'}^\dagger \cap \mathcal{R}^\dagger\Big).$$

We add a short note: (28) provides information only on the lim sup and the lim inf of sequences $\alpha^{p\dagger}(I_n^p)$ ($I_n^p \searrow \{\theta^p\}$), while (31) gives a stronger result: provided $\theta \in K_{\alpha,\alpha'}^\dagger \cap \mathcal{R}^\dagger$

$$\alpha^\dagger(I_n) \to \alpha \quad \text{as } I_n \searrow \theta \qquad \Rightarrow \qquad \alpha^{p\dagger}(I_n^\Psi) \to \alpha.$$

In addition, the type of argument given here does not apply to $F_\alpha^\dagger$ for this reason.

**vii) Dimension estimates** Let $A$ be a subset of $G_\alpha^\dagger \cap \mathcal{R}^\dagger$. According to v), we have $\Psi(A) \subset G_\alpha^{p\dagger} \cap \mathcal{R}^{p\dagger}$. We claim:

$$\dim(A) = \dim(\Psi(A)) \qquad \text{and} \qquad \mathrm{Dim}(A) = \mathrm{Dim}(\Psi(A)).$$

Take $\varepsilon \in (0, \alpha/2)$. Let $n$ be sufficiently large, i.e. $\phi(1/n^{\alpha-2\varepsilon}) \leq 1/n$ and $n \leq 2^{\varepsilon/(\alpha-2\varepsilon)}$. Set

$$A_n := \{\theta \in A : \Psi(\theta) \in I^p \text{ and } |I^p| \leq 1/n \text{ imply } \mu^{p\dagger}(I^p) \leq |I^p|^{\alpha-\varepsilon}\}$$

where $I^p$ denotes arbitrary intervals. Defining $I^\Psi$ as in (30)

$$A_{n,m} := \{\theta \in A_n : \theta \in I \text{ and } |I| \leq 1/m \text{ imply } |I^\Psi| \leq 1/n\}.$$

By continuity of $\Psi$ and v), $A = \cup_n A_n = \cup_{n,m} A_{n,m}$ ($A_n$ and $A_{n,m}$ are increasing in $n$ and $m$). For $n$ large enough, the estimate (29) applies to $I^\Psi$ for any interval $I$ of length $|I| \leq 1/m$ which intersects $A_{n,m}$. But (29) means that $\Psi$ is uniformly Lipschitz continuous on $A_{n,m}$ and preserves dimensions. Together with the $\sigma$-stability of Hausdorff and packing dimension the claim follows.

Since $\Psi$ is defined on $\mathbb{R}\backslash\mathcal{R}$ only, this argument might not seem trustworthy to the reader. This step being crucial to the whole proof, we proceed giving the details.

Consider a covering $\{I_j\}_j$ of $A_{n,m}$ by open intervals of length $|I_j| \leq \delta \leq 1/n$. Due to (28) and (29) $|I_j^\Psi| \leq 2|I_J|$, provided $I_j$ intersects $A_{n,m}$. We conclude,

$$\eta_\delta^\gamma(\Psi(A_{n,m})) \leq \sum_{j:\, I_j \cap A_{n,m} \neq \emptyset} |I_j^\Psi|^\gamma \leq 2^\gamma \cdot \sum_j |I_j|^\gamma$$

and

$$\eta_\delta^\gamma(\Psi(A_{n,m})) \leq 2^\gamma \cdot \eta_\delta^\gamma(A_{n,m}) \leq 2^\gamma \cdot \eta^\gamma(A).$$

Using $\sigma$-stability [Tr], we continue $\dim(\Psi(A)) \leq \sup_{n,m} \dim(\Psi(A_{n,m})) \leq \dim(A)$. The opposite inequality is trivial since $\Psi$ is one-to-one and expanding.

Finally, let $\{I_j^p\}_j$ be a packing of $\Psi(A_n)$ by open intervals of length $|I_j^p| \leq \delta \leq 1/n$. Consider $I_j := \langle \Psi^{-1}(I_j^p) \rangle$. First, each $I_j$ meets $A \subset G_\alpha^\dagger$ due to vi). Second, the $I_j$ are disjoint since $\Psi$ is one-to-one and only atoms $a$ with $L_a^p \subset I_j^p$ belong to $I_j$ by (24). Third, the last argument shows in addition that $I_j$ is open. Thus, $\{I_j\}_j$ forms a packing of $A_n$. Due to (28) and (29), we have $I_j^p \leq 2|I_j|$ and the rest follows by copying arguments of above and of Proposition 9.

**viii) The spectra** Again some notation. We apply the procedure described in i) to $\mu^{p\dagger}$. Let $\nu := \mu^{p\dagger}$ for the ease of notation. By construction, $\nu$ is a continuous measure on the $\theta^p$-axis. Its perturbation $\nu^p$ is, consequently, continuous and non–vanishing. In analogy to i), we consider its inverse measure $\nu^{p\dagger}$ as being defined on the $t^p$-axis.

Let $N(\theta^p) := M^{p\dagger}(\theta^p) = \nu([0, \theta^p])$ and $N^p(\theta^p) := \nu^p([0, \theta^p])$. The correspondence between points $N(\theta^p)$ on the $t$-axis and $N^p(\theta^p)$ on the $t^p$-axis is provided by an expansion $\chi$. As described in iv), we have $\chi(t) = N^p \circ N^{-1}(t)$, provided $N^{-1}(t)$ is a $\nu$-regular point. But all points of $\mathcal{R}^{p\dagger} = M^p(\mathcal{R})$ are certainly $\nu$-regular. In agreement with (23), we consider only the points of interest and set $\mathcal{S} := \mathcal{R}^{p\dagger}$.

By definition of $\mathcal{R}$, $M^p$ is a bijection between $\mathcal{R}$ and $\mathcal{R}^{p\dagger}$ with inverse $M^{p\dagger} = N$. Hence,

$$\chi(t) = N^p(M^p(t)) \qquad \text{for } t \in \mathcal{R}.$$

This expresses in a very clear picture how we distorted $\theta$- and $t$-space to get rid of gaps (by $M^p$) and atoms (by $N^p$) of the measure $\mu$. In analogy with iv), we let $\mathcal{S}^{p\dagger}$ denote the points of interest on the $t^p$-axis:

$$\mathcal{S}^{p\dagger} = N^p(\mathcal{S}) = N^p(\mathcal{R}^{p\dagger}) = N^p(M^p(\mathcal{R})) = \chi(\mathcal{R}).$$

Propositions 8 and 9, i.e. the inversion formula apply to the pair $\nu^p$ and $\nu^{p\dagger}$. We already know that — as far as the spectra are concerned — $\nu^p$ is 'close' to $\nu = \mu^{p\dagger}$ which again is close to $\mu^\dagger$. It remains to relate $\mu$ and $\nu^{p\dagger}$, more precisely, their 'sets of Hölder exponent' $K_{\alpha,\alpha'}[\mu]$ and $K_{\alpha,\alpha'}[\nu^{p\dagger}]$. Take $A \subset K_{\alpha,\alpha'}[\mu] \cap \mathcal{R}$. By iii) $A \subset K^p_{\alpha,\alpha'}[\mu] \cap \mathcal{R}$. By Lemma 2, $\mu^p$ and $\nu^\dagger$ coincide and $K_{\alpha,\alpha'}[\mu^p] = K_{\alpha,\alpha'}[\nu^\dagger]$. Applying vi) to $\nu$ yields $\chi(A) \subset K_{\alpha,\alpha'}[\nu^{p\dagger}] \cap \mathcal{S}^{p\dagger}$ with equality if $A = K_{\alpha,\alpha'} \cap \mathcal{R}$. It follows from vii) that $A$ and $\chi(A)$ have the same Hausdorff and packing dimension.

The inversion formula will provide us now with a dimension estimate of $N^{p\dagger} \circ \chi(A)$ where $N^{p\dagger}(t^p) := \nu^{p\dagger}([0, t^p])$. This, we would like to compare with the dimension of $M(A)$. By (10), $N^{p\dagger}$ is bijective with inverse $N^p$. We conclude that $N^{p\dagger} \circ \chi = N^{p\dagger} \circ N^p \circ M^p = M^p$, and, for $t \in \mathcal{R}$:

$$N^{p\dagger} \circ \chi = M^p = \Psi \circ M. \tag{32}$$

The 'diagram' commutes. In other words, the distortions of the $t$ and the $\theta$ axis 'match'. Furthermore, the inverse of $\nu^{p\dagger}$ is $\nu^p$ by Lemma 2, and (32) shows that $M^p(A)$ is a subset of $K_{\alpha,\alpha'}[\nu^p] \cap \mathcal{S}$. Again, we have equality if $A = K_{\alpha,\alpha'} \cap \mathcal{R}$.

Finally, $M(A)$ and $M^p(A)$ have the same dimensions by vii). Furthermore, $\Psi^{-1}$ is well defined on all of $M^p(A)$ due to $\mathcal{R}^{p\dagger} = \Psi(\mathcal{R}^\dagger)$. Thus, $M(A) \subset K^\dagger_{1/\alpha',1/\alpha} \cap \mathcal{R}^\dagger$ with equality if $A = K_{\alpha,\alpha'} \cap \mathcal{R}$.

Using the results of the three preceding paragraphs, and applying Proposition 8 to the measure $\nu^{p\dagger}$, our chain of estimates reads:

$$\dim(A) = \dim(\chi(A)) \geq \alpha \cdot \dim(N^{p\dagger} \circ \chi(A)) = \alpha \cdot \dim(\Psi \circ M(A)) = \alpha \cdot \dim(M(A)).$$

Similarly,

$$\dim(A) \geq \alpha \cdot \dim(M(A)) \geq \alpha/\alpha' \cdot \dim(A)$$

and

$$\mathrm{Dim}(A) \leq \alpha' \cdot \mathrm{Dim}(M(A)) \leq \alpha'/\alpha \cdot \mathrm{Dim}(A).$$

This is the strongest result available with the arguments given here. As already mentioned, we loose details on the 'Hölder analysis' by mapping with $\Psi$ and $\chi$. In particular, the only accumulation points of $\alpha(I)$ (as $I \searrow \{t\}$) which are preserved are the limsup and the liminf. On the other hand, we need information on both of these accumulation points since $G_{1/\alpha}[\nu^p] = F_\alpha[\nu^{p\dagger}]$. ♢

As an immediate consequence of step viii) above, we have

**Corollary 23** *Let $\mu$ be a probability measure on $[0,1]$ and let $\mathcal{R}$ denote its regular points. Then*

$$M(K_{\alpha,\alpha'} \cap \mathcal{R}) = K^\dagger_{1/\alpha',1/\alpha} \cap \mathcal{R}^\dagger.$$

*In other words, for all but countably many $t$ the following equivalence holds: $\alpha(t) = \alpha$ if and only if $\alpha^\dagger(M(t)) = 1/\alpha$.*

**Final Remark** In the case of (discontinuous) self-similar measures, the explicit construction of the measures $\phi_q$ with $\phi_q(K_\alpha) = 1$ (Theorem 16) implies that $K_\alpha$ is of full $f(\alpha)$ -dimensional Hausdorff and packing measure. In other words, the inversion formula is sharp for self-similar measures in the sense of giving 'exact dimensions'. It would be interesting to know whether this is true in general.

With the notion of discontinuous self-similar measures a new family of multifractals have been introduced. While generalizations of self-similarity to infinite number of copies and to randomly picked maps result in concave spectra, we find here for the first time self-similar measures with non-concave fine multifractal spectra. So far, non-concave spectra were known only for non-multiplicative measures where $K_\alpha$ is no longer dense in the support of the measure.

*Acknowledgment.* We are very grateful to Prof. Jacques Peyrière for helpful discussions. RHR gratefully acknowledges financial support by the ONR, grant N00014-90.J.1026.

## References

[AP] M. Arbeiter and N. Patzschke, Self-Similar Random Multifractals, *Math. Nachr.* **181** (1996) pp 5–42

[BMP] G. Brown, G. Michon and J. Peyrière, On the Multifractal Analysis of Measures, *J. Stat. Phys.* **66** (1992) pp 775–790

[CM] R. Cawley & R. D. Mauldin, Multifractal Decompositions of Moran Fractals, *Adv. in Math.* **92** (1992) pp 196–236

[CLP] P. Collet, J. Lebovitc and A. Porcio, The Dimension Spectrum of Some Dynamical Systems, *J. Stat. Phys.* **47** (1987) pp 609–644.

[E] R. Ellis, Large Deviations for a General Class of Random Vectors, *Ann. Prob.* **12** (1984) pp 1–12

[F] K. J. Falconer, 'Fractal Geometry: Mathematical Foundations and Applications' John Wiley and Sons, New York (1990)

[F2] K. J. Falconer, The Multifractal Spectrum of Statistically Self-Similar Measures, *J. Theor. Prob.* **7** (1994) pp 681–702

[H] J. Hutchinson, Fractals and Self Similarity, *Indiana Univ. Math. J.* **30** (1981) pp 713–747

[LN] K.-S. Lau and S.-M. Ngai, Multifractal Measures and a Weak Separation Condition, *Adv. Math.* accepted for publication (1995).

[LV] J. Levy-Vehel and R. Vojak, Multifractal analysis of Choquet capacities: Preliminary Results, *Adv. Appl. Math.* accepted for publication (1995).

[KP] J.-P. Kahane and J. Peyrière, Sur Certaines Martingales de Benoit Mandelbrot, *Adv. Math.* **22** (1976) pp131–145

[M] B. B Mandelbrot, "The Fractal Geometry of Nature", New York, Freeman (1982).

[M90] B. B Mandelbrot, New 'Anomalous' Multiplicative Multifractals: Left-Sided $f(\alpha)$ and the Modelling of DLA, *Physica* **A 168**, 1990, pp 95–111

[MEH] B. B Mandelbrot, C. J. G. Evertsz and Y. Hayakawa, Exactly Self-Similar Left Sided Multifractal Measures, *Physical Review A* **42** 8 (1990) pp 4528–4536

[MR] B. B Mandelbrot and R. H. Riedi, Inverse Measures, the Inversion formula and Discontinuous Multifractals, *Adv. Appl. Math.* **18** (1997) pp 50–58.

[Mor] P. A. P. Moran, Additive Functions of intervals and Hausdorff Measure, *Proc. Cambr. Phil. Soc.* **42** (1946) pp 15–23.

[O] L. Olsen, Random geometrically graph directed self-similar multifractals, *Pitman Research Notes Math. Ser.* **vol 307** (1994)

[PR] R. Pastor-Satorras and R. H. Riedi, Numerical Estimates of Generalized Dimensions $D(q)$ for Negative $q$, *J. Phys. A: Math. Gen.* **29** (1996) L391–L398

[Pa] N. Patzschke, Strong Open Set Condition in the Random Case submitted *Proc. AMS* 1996

[PW] Y. Pesin and H. Weiss, A Multifractal Analysis of Equilibrium Measures for Conformal Expanding Maps and Moran-like Geometric Constructions, *J. Stat. Phys.* **86** (1997) 233–275

[Ra] D. A. Rand, The Singularity Spectrum $f(\alpha)$ for Cookie-Cutters, *Erg. Th. Dyn. Sys.* **9** (1989) pp 527–541

[R1] R. H. Riedi, An Improved Multifractal Formalism and Self-Similar Measures, *J. Math. Anal. Appl.* **189** (1995) pp 462–490

[R2] R. H. Riedi, Seven Definitions of Multifractal Scaling Exponents, *forthcoming*

[RM1] R. H. Riedi and B. B Mandelbrot, Multifractal Formalism for Infinite Multinomial Measures, *Adv. Appl. Math.* **16** (1995) pp 132–150

[RM2] R. H. Riedi and B. B Mandelbrot, The Inversion Formula for Continuous Multifractals, *Adv. Appl. Math.* **19** (1997) pp 332–354

[Sch] A. Schief, Separation properties for self-similar sets, *Proc. Amer. Math. Soc.* **122** (1994), pp 111–115

[S] R. Strichartz Self-Similar Measures and their Fourier Transforms III *Indiana Univ. Math. J.* **42** (1993) pp 367–411

[Tr] C. Tricot, Two Definitions of Fractional Dimension, *Math. Proc. Cambr. Phil. Soc.* **91** (1982) pp 57–74.