Explicit Bounds for the Hausdorff Dimension of Certain Self-Affine Sets

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Abstract

A lower bound of the Hausdorff dimension of certain self-affine sets is given. Moreover, this and other known bounds such as the box dimension are expressed in terms of solutions of simple equations involving the singular values of the affinities.

Keyword Codes: G.2.1;G.3
Keywords: Discrete Mathematics, Combinatorics; Probability and Statistics

1 INTRODUCTION

A compact set K in Euclidean space $\mathbb{R}^d$ such as the middle third Cantor set may carry a rich geometrical structure. In order to measure the complexity of its geometry, the box dimension and the Hausdorff dimension have been introduced [4]:

$$d_{\rm box}(K) := \lim_{\delta\downarrow 0} \frac{\log \mathcal{N}_\delta(K)}{-\log\delta},$$

where $\mathcal{N}_\delta(K)$ is the minimal number of balls of radius δ needed to cover K, and

$$d_{\rm HD}(K) := \inf\{\alpha \geq 0 \,:\, m^\alpha(K) = 0\} = \sup\{\alpha \geq 0 \,:\, m^\alpha(K) = \infty\},$$

where m^α denotes the α-dimensional Hausdorff measure

$$m^\alpha(E) := \sup_{\delta>0} \inf \{ \sum_{l=1}^{\infty} ({\rm diam}(S_l))^\alpha \,:\, E \subset \bigcup_{l=1}^{\infty} S_l,\ {\rm diam}(S_l) \leq \delta \ \ \forall l \in \mathbb{N}\}.$$

After it was discovered that Hausdorff and box dimension of self-similar sets coincide [7], great effort has been made to calculate the dimensions of self-affine sets [8, 6, 12, 1, 3, 5, 9]. Though in this case the two dimensions coincide at least ‘almost surely’ [3], the explicitly solvable cases mostly turn out to be exceptional, yielding values which differ from the expected answer [8, 6]. So, looking for ‘sure’ results, one is often forced to content oneself with bounds, as in [5], and as shall be done here.

We will consider self-affine sets K which arise from an iterated function system in the following way. Assume that the Euclidean space is split into two fixed orthogonal subspaces: $\mathbb{R}^d = \mathbb{R}^{d'} \oplus \mathbb{R}^{d''}$. (For a fractal surface take $d' = 2$ and $d'' = 1$.) Fix a natural number r and consider r affine transformations w_i of $\mathbb{R}^d$ which reduce to similarities in the subspaces $\mathbb{R}^{d'}$ and $\mathbb{R}^{d''}$, i.e.

$$w_i \,:\, \mathbb{R}^d = \mathbb{R}^{d'} \oplus \mathbb{R}^{d''} \to \mathbb{R}^d \qquad (x, y) \mapsto (\lambda_i \Theta_i x, \nu_i \Xi_i y) + (u_i, v_i) \tag{1}$$

where Θ_i and Ξ_i are orthogonal transformations, u_i and v_i are from $\mathbb{R}^{d'}$ and $\mathbb{R}^{d''}$ respectively, and where the ratios λ_i and ν_i of the similarities satisfy

$$0 < \nu := \min\{\lambda_1, \ldots, \lambda_r, \nu_1, \ldots, \nu_r\} \leq \lambda := \max\{\lambda_1, \ldots, \lambda_r, \nu_1, \ldots, \nu_r\} < 1.$$

We will call $w_1, \ldots, w_r$ a family of *diagonal affine contractions*.

It is well known [7] that there exists a unique nonempty compact set satisfying

$$K = \bigcup_{i=1}^{r} w_i(K). \tag{2}$$

It is the aim of this paper to provide a lower bound Γ of the Hausdorff dimension of K, which holds under a certain open set condition. This condition as well as some geometrical lemmas are given in section 2. The bound Γ is elaborated in section 3 using limit theorems from probability theory. See theorem 4. In section 4, this bound is compared with the lower bound d_- given in [5], with the ‘almost sure’ value of $d_{\mathrm{HD}}(K)$ [3] and with the actual box dimension of K, for which explicit formulas are provided.

2 GEOMETRICAL RESULTS

First, the geometrical situation we will deal with will be made precise and the formalism of symbolic dynamics is introduced. Then, two geometrical lemmas are given, which provide lower bounds of $d_{\mathrm{HD}}(K)$.

First, assume the existence of a nonempty, bounded, connected, open set O satisfying the so-called open set condition (OSC) [7]

$$w_i(O) \subset O \quad (i = 1, \ldots, r) \qquad \text{and} \qquad w_i(O) \cap w_j(O) = \emptyset \quad (i \neq j). \tag{3}$$

Let

$$V_{\mathrm{nil}} := \overline{O} \qquad \text{and} \quad V_{\underline{i}} := w_{\underline{i}}(\overline{O}) := w_{i_1} \circ \ldots \circ w_{i_n}(\overline{O}).$$

Thereby we introduced the empty word nil and the words of length $|\underline{i}| := n$, $(n \in \mathbb{N})$,

$$\underline{i} := i_1 \ldots i_n \,\in I_n := \{1, \ldots, r\}^n.$$

Moreover, we set $I := \cup I_n$, $\underline{i} * k := i_1 \ldots i_n k$ and $\underline{i}|m := i_1 \ldots i_m$ for $m \leq |\underline{i}|$. Since $V_{\underline{i}*k} = w_{\underline{i}*k}(\overline{O}) = w_{\underline{i}}(V_k) \subset w_{\underline{i}}(\overline{O}) = V_{\underline{i}}$, the sequence K_n

$$K_n := \bigcup_{\underline{i} \in I_n} V_{\underline{i}} \quad \text{decreases to} \quad K = \bigcap_{n \in \mathbb{N}} K_n. \tag{4}$$

While in the case of similarities the OSC is enough to calculate $d_{\text{box}}(K)$ and $d_{\text{HD}}(K)$ [7], one more regularity condition is needed here: Denote by R the smallest closed rectangle containing O and with sides parallel to the axes. For the sake of simplicity assume $R = [0,1]^2$, which is not really a restriction as far as dimensions are concerned. The additional hypothesis on O is: There is a $\varrho > 0$, i and j with $1 \le i \le d' < j \le d$ and ξ, x, ζ, z from $\mathbb{R}^d$, such that the points

$$\begin{array}{lcl} (\xi_1,\ldots,\xi_{i-1},t,\xi_{i+1},\ldots,\xi_d) & , & (x_1,\ldots,x_{i-1},1-t,x_{i+1},\ldots,x_d), \\ (\zeta_1,\ldots,\zeta_{j-1},t,\zeta_{j+1},\ldots,\zeta_d) & , & (z_1,\ldots,z_{j-1},1-t,z_{j+1},\ldots,z_d) \end{array} \tag{5}$$

belong to O for all $t \in]0,\varrho[$. Loosely speaking, it is possible to cross R corresponding to the two invariant subspaces on two piecewise linear paths which are parallel to the axes and which do not leave O. Any set O with the above properties is called *round open set*.

Definition 1 *We will say that K is* diagonal self-affine *iff it is the nonempty, compact and invariant (2) set of a family $(w_1,\ldots,w_r)$ of diagonal affine contractions with a round open set. (The singular values of w_i will be denoted by λ_i and ν_i).*

In order to estimate the Hausdorff measure of K, a certain collection of sets $(V_{\underline{i}})_{\underline{i}\in J_\delta}$ with 'width' approximately equal to δ is useful. For any finite word $\underline{i} = i_1 \ldots i_n$ let

$$\lambda_{\underline{i}} := \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_n} \qquad \nu_{\underline{i}} := \nu_{i_1} \cdot \ldots \cdot \nu_{i_n} \tag{6}$$

and

$$\kappa(\underline{i}) := \min(\lambda_{\underline{i}}, \nu_{\underline{i}}) \ge \kappa(i_1 \ldots i_{n-1}) \cdot \kappa(i_n) \ge \kappa(i_1) \cdot \ldots \cdot \kappa(i_n).$$

Since κ is only sub-multiplicative we prefer the slightly different notation and won't write $\kappa_{\underline{i}}$. Trivially $\kappa(\underline{i}) \le \lambda^n \to 0$ $(n \to \infty)$. For any $\delta \in]0,\nu[$ set

$$J_\delta := \{\underline{i} = i_1 \ldots i_n \in I \,:\, \kappa(\underline{i}) \le \delta < \kappa(i_1 \ldots i_{n-1})\}. \tag{7}$$

The length of any word of J_δ amounts at most $m_\delta := \lceil \log \delta / \log \lambda \rceil$. On the other hand, for any $\underline{j}$ from I_{m_δ} there is a unique n such that $j_1 \ldots j_n \in J_\delta$, since $\kappa(\underline{j}) \le \delta$ and $\kappa(j_1 \ldots j_m) \le \lambda \kappa(j_1 \ldots j_{m-1})$. Consequently, J_δ is *secure* [7], i.e.

$$K \subset \bigcup_{\underline{i}\in J_\delta} V_{\underline{i}}, \tag{8}$$

and *tight* [7], i.e. for any different words $\underline{i} \ne \underline{j}$ from J_δ there is $k \le \min(|\underline{i}|,|\underline{j}|)$ with $i_k \ne j_k$, hence, with (3),

$$w_{\underline{i}}(O) \cap V_{\underline{j}} = \emptyset \tag{9}$$

Finally,

$$\nu\delta \le \kappa(\underline{i}) \le \delta \quad \text{for all } \underline{i} \in J_\delta. \tag{10}$$

Next, a lemma is required, saying that a set of size δ is not intersected by too many sets $V_{\underline{i}}$ with $\underline{i} \in J_\delta$. It is only here, where the 'roundness' of O is actually needed.

Lemma 2 *Given diagonal affinities $w_1,\ldots,w_r$ and a round open set O, there is a number b such, that $\#\{\underline{i} \in J_\delta \,:\, V_{\underline{i}} \cap W \ne \emptyset\} \le b$ for all $\delta > 0$ and for all balls W of radius δ.*

Proof For simplicity we give the proof for the case $d' = d'' = 1$. The general case can be treated with the same argumentation.

Let $\delta > 0$ and let W be a ball of radius δ with centre (x, y). Set $W' := [x-2\delta, x+2\delta]\times[y-2\delta, y+2\delta]$. We will only be concerned with words $\underline{i}$ s.t. $\lambda_{\underline{i}} \geq \nu_{\underline{i}}$, leading to a bound b^+. By symmetry, a bound b^- will be obtained for the words with $\lambda_{\underline{i}} \leq \nu_{\underline{i}}$, and $b = b^+ + b^-$ will be enough.

Since O is connected and bounded, there is a path γ within O joining $Q_0 := (0, y_1)$ with $Q_N := (1, y_2)$, and which consists of finite many straight line segments, each one parallel to one of the axes. Of course it is possible to choose $N \geq 2$. Enumerate the vertices of γ by Q_l $(l = 0, \ldots, N)$, and choose $\varrho' > 0$ such that the balls $U(\varrho', Q_l)$ are contained in O for $l = 1, \ldots, N-1$. Consequently, the set $w_{\underline{i}}(O)$ contains the ellipses $w_{\underline{i}}(U(\varrho', Q_l))$ and hence the balls $U_{\underline{i},l} := U(\varrho'\nu\delta, w_{\underline{i}}(Q_l))$ due to (10). Take $\underline{i} \neq \underline{j}$ from J_δ. By (9), the balls $U_{\underline{i},l}$ must be disjoint with $U_{\underline{j},m}$ whenever $1 \leq l, m \leq N-1$. This is one main point of the proof.

Now consider the words $\underline{i}$ from J_δ, for which $V_{\underline{i}}$ meets W. Two types will be distinguished. Words $\underline{i}$ of the first type have a large part of a ball $U_{\underline{i},l}$ $(1 \leq l \leq N-1)$ lying in W'. Since $U_{\underline{i},l}$ and W' are of nearly equal size, there can't be too many words of this type. Words $\underline{i}$ of the second type have all $U_{\underline{i},l}$ $(1 \leq l \leq N-1)$ outside W'. Since $V_{\underline{i}}$ meets W, the paths $w_{\underline{i}}(\gamma)$ must intersect the boundary of W'. Since they are 'parallel' they cannot come too close due to the disjoint balls $U_{\underline{i},l}$, and their number is bounded too.

Type 1): There is $l \neq 0, N$ such that $w_{\underline{i}}(Q_l)$ lies in W'. Then W' contains at least one quarter of the ball $U_{\underline{i},l}$. Using disjointness and comparing volumes, the number of words of type 1) is seen to be bounded by

$$b_1 := \frac{64}{\pi(\varrho'\nu)^2}.$$

Type 2): $R_{\underline{i}}$ is a rectangle which meets W and with sides $\lambda_{\underline{i}}$ and $\nu_{\underline{i}}$. Since $\nu_{\underline{i}} = \kappa(\underline{i}) \leq \delta$ and since $\underline{i}$ is not of type 1), the path $w_{\underline{i}}(\gamma)$ joining 'left' and 'right' end of $R_{\underline{i}}$ must contain a point of the form $S_{\underline{i}} = (x \pm 2\delta, y_{\underline{i}})$. Denote by $h_{\underline{i}}$ the horizontal part of $w_{\underline{i}}(\gamma)$, which contains $S_{\underline{i}}$. Since $\underline{i}$ is not of type 1) and since $N \geq 2$, there is $l \neq 0, N$ such that $w_{\underline{i}}(Q_l)$ is an end point of $h_{\underline{i}}$. This point lies outside W' but in the interior of $V_{\underline{i}}$. Take a word $\underline{j}$ of type 2) different from $\underline{i}$. Since the ball $U_{\underline{i},l}$ is disjoint with $V_{\underline{j}}$ by (9), it cannot intersect $h_{\underline{j}}$. Vice versa, $U_{\underline{j},m}$ cannot meet $h_{\underline{i}}$. Consequently, $S_{\underline{i}}$ and $S_{\underline{j}}$ are at least at distance $\nu\delta\varrho'$ of each other. Comparing the length of $\partial W'$ with these distances shows, that at the most

$$b_2 := 2\left(\frac{4}{\varrho'\nu} + 1\right)$$

words of type 2) are possible. Thus $b^+ := b_1 + b_2$ is enough. ◇

Since any set $V_{\underline{i}}$ contains a ball of radius $\mathrm{const} \cdot \kappa(\underline{i})$, it is natural to estimate $m^\gamma(K)$ from below by certain $\sum \kappa(\underline{i})^\gamma$. For any set of words J let

$$\sigma(\gamma, J) := \sum_{\underline{i} \in J} \kappa(\underline{i})^\gamma = \sum_{\underline{i} \in J^+} \nu_{\underline{i}}{}^\gamma + \sum_{\underline{i} \in J^-} \lambda_{\underline{i}}{}^\gamma, \tag{11}$$

where

$$J^+ := \{\underline{i} \in J \,:\, \lambda_{\underline{i}} > \nu_{\underline{i}}\} \qquad J^- := \{\underline{i} \in J \,:\, \lambda_{\underline{i}} \leq \nu_{\underline{i}}\}. \tag{12}$$

Lemma 3 *Let K be a diagonal self-affine set. Assume that N and γ are such that $\sigma(\gamma, I_N) \geq 1$. Then $d_{\mathrm{HD}}(K) \geq \gamma$.*

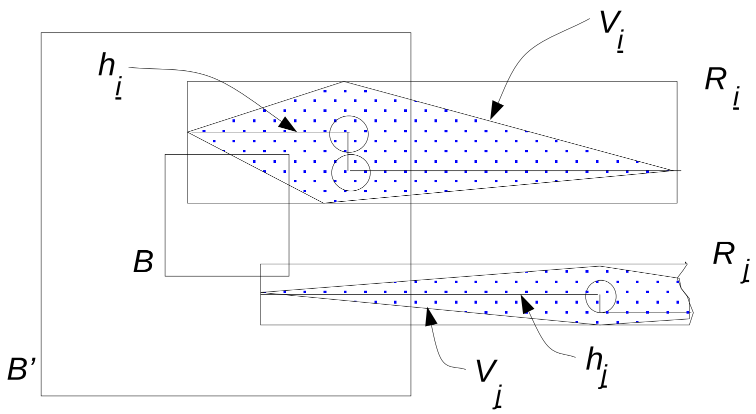

Figure 1: The number of rectangles $R_{\underline{i}}$ with $\lambda_{\underline{i}} \geq \nu_{\underline{i}} \simeq \delta$ meeting a ball W of radius δ (or a box of size δ for that matter) is bounded due to the disjoint balls $U_{\underline{i},l}$ and the horizontal parts $h_{\underline{i}}$.

Proof i) Note that K is also invariant under the set of affinities $(w_{\underline{i}})_{|\underline{i}|=N}$. Thus, performing a natural change in the encoding of K, i.e. $I_N = \{1,\ldots,r'\} =: I_1'$ with $r' = r^N$, we may assume without loss of generality that $N=1$. Furthermore we may assume that $\gamma \geq 0$.

ii) Take an arbitrary cover $(S_l)_{l\in\mathbb{N}}$ of K. For $S_l \neq \emptyset$ let W_l be an open ball of radius $\delta_l := 2\cdot\mathrm{diam}(S_l)$ centred in a point of S_l. By compactness, K is covered by a finite subcollection of the so defined W_l, say $W_1,\ldots,W_p$. Moreover, since $U := W_1\cup\ldots\cup W_p$ is open, a compactness argument gives an integer q such that U even covers K_n (4) for all $n\geq q$.

iii) Now consider $H_l := \{\underline{i}\in J_{\delta_l} : V_{\underline{i}}\cap W_l \neq \emptyset\}$. By lemma 2 there is a number b depending neither on l nor on the cover S_l with $\#H_l \leq b$. Consequently, with (10),

$$\sum_{l=1}^{\infty}\mathrm{diam}(S_l)^\gamma \geq 2^{-\gamma}\sum_{l=1}^{p}\delta_l{}^\gamma \geq 2^{-\gamma}\sum_{l=1}^{p}\frac{1}{b}\sum_{\underline{i}\in H_l}\kappa(\underline{i})^\gamma \geq (2^\gamma b)^{-1}\cdot\sigma(\gamma,H),$$

where H denotes the union of the H_l. The final two steps of the proof show that H is secure and that $\sigma(\gamma,L)\geq 1$ for any finite, secure set L. Consequently, $m^\gamma(K)\geq 1/(2^\gamma b)>0$ and $d_{\mathrm{HD}}(K)\geq\gamma$.

iv) By making q larger if necessary we may assume that every word of H is at most of length q (since $\#H\leq bp$) and that $\lambda^q\leq\delta_l$ for $l=1,\ldots,p$. To prove that H is secure it is enough to show: for any $\underline{j}\in I_q$ exists an integer n with $\underline{j}|n\in H$. Take $\underline{j}$ from I_q. Since U covers K_q, there is l with $V_{\underline{j}}\cap W_l\neq\emptyset$. Since $\kappa(\underline{j})\leq\lambda^q\leq\delta_l$ there is by (7) a number n such that $\underline{i} := \underline{j}|n\in J_{\delta_l}$. Finally, $V_{\underline{i}}\supset V_{\underline{j}}$ implies $\underline{i}\in H_l\subset H$ and the claim follows.

v) Let L be any finite, secure set. We show that $\sigma(\gamma,L)\geq 1$. First, let $L_1 := \{\underline{i}\in L : \underline{i}|m\notin L\ \forall m<|\underline{i}|\}$. By definition, L_1 is tight. Since only extensions of other words contained in L have been thrown away, L_1 is secure. Obviously, $\sigma(\gamma,L)\geq\sigma(\gamma,L_1)$. Now, to prove the claim consider the following inductive process which generates a 'shrinking' sequence of tight and secure sets L_m: Take a word $\underline{i}=i_1\ldots i_n\in L_m$ with maximal length. Assume that $n\geq 2$. Since L_m is tight and secure, it must contain all the words $i_1\ldots i_{n-1}k$ $(k=1,\ldots,r)$. Replacing these r words by their 'predecessor' $i_1\ldots i_{n-1}$ yields a new set L_{m+1} which is still secure and tight. If $L_m = I_1$, then set $L_{m+1}=L_m$.

This defines the process. Consider the sequence L_m. One has $\sigma(\gamma, L_m) \geq \sigma(\gamma, L_{m+1})$ since

$$\sum_{k=1}^{r} \kappa(\underline{j} * k)^\gamma \geq \sum_{k=1}^{r} \kappa(\underline{j})^\gamma \kappa(k)^\gamma = \kappa(\underline{j})^\gamma \sigma(\gamma, I_1) \geq \kappa(\underline{j})^\gamma. \tag{13}$$

By induction $\sigma(\gamma, L) \geq \sigma(\gamma, L_m)$. Moreover, the number of words in L_m decreases strictly in m unless $L_m = I_1$. Since L_1 is finite, this implies $L_m = I_1$ for m large enough. By assumption $\sigma(\gamma, I_1) \geq 1$ which completes the proof. For further use, note that $\sigma(\gamma, I_k) \geq 1$ since I_k is secure and finite. ◇

Remark The argumentation above is of purely geometrical kind. In fact, it provides an alternative proof of Moran's theorem [7].

3 THE MAIN RESULT

Now, the lower bound of $d_{\mathrm{HD}}(K)$ given by lemma 3 shall be optimized. We denote by γ_n the unique (positive) numbers satisfying $\sigma(\gamma_n, I_n) = 1$. With (13) it is easy to see that $\sigma(\gamma_n, I_{kn}) \geq 1$. Since $\sigma(\gamma, I_n)$ is strictly decreasing in γ one finds $\gamma_n \leq \gamma_{kn}$ and thus $\sup(\gamma_n) = \limsup \gamma_n$. On the other hand, $\lim \gamma_n = \Gamma$ as will be shown below. Hence, Γ is the optimal lower bound of $d_{\mathrm{HD}}(K)$ which can be extracted from lemma 3. Ex. 2 shows that there may be no better bound on $d_{\mathrm{HD}}(K)$ unless it involves the translations (u_i, v_i).

In order to give the value of Γ, let γ^+ resp. γ^- be the unique numbers satisfying

$$\sum_{i=1}^{r} \nu_i^{\gamma^+} = 1 \quad \text{resp.} \quad \sum_{i=1}^{r} \lambda_i^{\gamma^-} = 1. \tag{14}$$

Provided there are $i \neq j$ with

$$\lambda_i < \nu_i \quad \text{and} \quad \lambda_j > \nu_j, \tag{15}$$

denote by (t_0, γ_0) the unique solution of

$$\left| \begin{array}{rcl} \sum_{i=1}^{r} \nu_i^\gamma (\lambda_i/\nu_i)^t & = & 1 \quad (a) \\ \sum_{i=1}^{r} \log(\lambda_i/\nu_i) \nu_i^\gamma (\lambda_i/\nu_i)^t & = & 0 \quad (b) \end{array} \right| \tag{16}$$

(The existence of (t_0, γ_0) will be shown below.) Otherwise, i.e. if (15) does not hold, set $\gamma_0 = 0$. Finally, let

$$\Gamma^+ := \begin{cases} \gamma^+ & \text{if } \sum_{i=1}^{r} \log(\lambda_i/\nu_i) {\nu_i}^{\gamma^+} \geq 0, \\ \gamma_0 & \text{otherwise,} \end{cases} \qquad \Gamma^- := \begin{cases} \gamma^- & \text{if } \sum_{i=1}^{r} \log(\nu_i/\lambda_i) {\lambda_i}^{\gamma^-} \geq 0, \\ \gamma_0 & \text{otherwise.} \end{cases}$$

Theorem 4 *Let K be a diagonal self-affine set. Then*

$$\boxed{d_{\mathrm{HD}}(K) \geq \Gamma := \max(\Gamma^+, \Gamma^-).}$$

The first steps of the proof explain the definition of Γ and provide some notation.

Proof Remind that $d_{\mathrm{HD}}(K) \geq \sup \gamma_n$ by lemma 3, where $\sigma(\gamma_n, I_n) = 1$.

o) Assume first that $\lambda_i \geq \nu_i$ $(i = 1, \ldots, r)$. Then one finds

$$\sigma(\gamma, I_n) = \Big(\sum_{i=1}^{r} \kappa(i)^\gamma\Big)^n.$$

Thus, all γ_n then coincide with γ^+, and hence with Γ by direct verification. The assertion follows immediately, and a similar argument holds for the case $\nu_i \geq \lambda_i$ $(i = 1, \ldots, r)$. Thus assume (15) for the remainder.

i) Consider the probability space $(I_\infty, \mathcal{B}, P)$ where $I_\infty := \{i = 1, \ldots, r\}^{\mathbb{N}}$ is endowed with the product topology, where $\mathcal{B}$ is the σ-algebra of its Borel sets and where P is the product measure on $\mathcal{B}$ induced by the measures

$$\{i\} \mapsto \frac{{\nu_i}^\gamma}{\sum\limits_{i=1}^{r} \nu_i^\gamma}$$

on the factors $\{1, \ldots, r\}$ of I_∞. Note that P depends on γ. The random variables

$$X_n \,:\, I_\infty \to \mathbb{R} \quad (i_1, i_2, \ldots) \mapsto \log(\lambda_{i_n}/\nu_{i_n})$$

are independent and identically distributed due to the property of the product measure. Set $Z_n := X_1 + \ldots + X_n$. Then, for any fixed γ,

$$\sigma_n^+(\gamma) := \sum_{\underline{i} \in I_n^+} {\nu_{\underline{i}}}^\gamma = \Big(\sum_{i=1}^{r} \nu_i^\gamma\Big)^n \cdot P[Z_n > 0]. \tag{17}$$

Provided the expectation $E[X_n]$ is nonnegative, the Law of Large Numbers implies that γ^+ still rules the asymptotical behaviour of σ_n^+ in a way made precise in step iv). Otherwise, the moment generating function

$$M(t) := E[e^{tX_n}] = \Big(\sum_{i=1}^{r} \nu_i^\gamma\Big)^{-1} \sum_{i=1}^{r} (\lambda_i/\nu_i)^t {\nu_i}^\gamma$$

is involved: Provided $P[X_n > 0] > 0$ and $E[X_n] < 0$, one has

$$\lim_{n\to\infty} \frac{1}{n} \log P[Z_n > 0] = \log \inf_t M(t) \tag{18}$$

by Chernoff's theorem [2, p 147]. As will be shown, t_0 minimizes $M(t)$ by (16 b), while (16 a) combines (17) and (18). Thus, the asymptotical behaviour of σ_n^+ is then ruled by γ_0. This explains the definition of Γ.

ii) Next, the solvability of (16) has to be established. For convenience

$$\chi(\gamma, t) := \sum \nu_i^\gamma (\lambda_i/\nu_i)^t.$$

For fixed γ, (16 b) has a unique solution $t_0 = t_0(\gamma)$ due to (15). Obviously, $M'(t_0) = 0$ and t_0 minimizes M. Moreover, t_0 depends continuously differentiable on γ since $\chi_{.tt} > 0$. The function of interest for (17) is

$$h(\gamma) := \sum_{i=1}^{r} \nu_i^\gamma \cdot M(t_0(\gamma)) = \chi(t_0(\gamma), \gamma),$$

which is strictly decreasing by the following argument: $\chi_{.t}(t_0,\gamma)$ vanishes by definition of t_0 and thus

$$h'(\gamma) = \chi_{.t}\cdot \tfrac{\partial}{\partial\gamma}t_0 + \chi_{.\gamma} = \sum_{i=1}^{r}\log\nu_i\cdot{\nu_i}^{\gamma}(\lambda_i/\nu_i)^{t_0} \le \log\lambda\cdot h(\gamma) < 0.$$

The mean value theorem implies $h(\gamma)\to\infty$ ($\gamma\to-\infty$). On the other hand, $M(t_0)\le M(0)=1$ for all γ. As a first consequence, $h(\gamma)\le\sum\nu_i^\gamma\to 0$ ($\gamma\to\infty$). This establishes the existence and the uniqueness of γ_0 with $h(\gamma_0)=1$, which is (16 a). As a second consequence, it implies $h(\gamma^+)\le\chi(0,\gamma^+)=1=h(\gamma_0)$ and hence $\gamma_0\le\gamma^+$.

iii) The asymptotical behaviour of σ_n^+ is best described in the notation of ii). Fix γ. If $E[X_n]=\chi_{.t}(0,\gamma)/\chi(0,\gamma)\ge 0$, i.e. $\chi_{.t}(0,\gamma)\ge 0$, then $P[Z_n>0]\ge P[(Z_n-nE)/(n\sqrt{\mathrm{var}})>0]\to 1/2$. If $E[X_n]<0$, then Chernoff's theorem (18) can be applied due to (15). This leads with (17) to

$$l^+(\gamma) := \lim_{n\to\infty}\frac{1}{n}\log\sigma_n^+(\gamma) = \begin{cases}\log\chi(0,\gamma) & \text{if } \sum_{i=1}^{r}\log(\lambda_i/\nu_i){\nu_i}^{\gamma}\ge 0\\ \log h(\gamma) & \text{otherwise.}\end{cases}$$

iv) Finally, we show how Γ^+ rules the asymptotics of σ_n^+:

$$l^+(\gamma)>0 \quad\text{if}\quad \gamma<\Gamma^+ \qquad\text{and}\qquad l^+(\gamma)<0\quad\text{if}\quad\gamma>\Gamma^+.$$

From $M(0)=1$ follows $\chi(0,\gamma)\ge h(\gamma)$ thus $l^+(\gamma)>0$ for all $\gamma<\gamma_0$, resp. $l^+(\gamma)<0$ for all $\gamma>\gamma^+$. It remains to consider $\gamma\in[\gamma_0,\gamma^+]$. Assume $\chi_{.t}(0,\gamma)=0$. Then $t_0(\gamma)=0$ and $h(\gamma)=\chi(0,\gamma)$. But since h and $\chi(0,\cdot)$ are both strictly monotonous decreasing, and since $h(\gamma_0)=1=\chi(0,\gamma^+)$, this implies $\gamma^+=\gamma_0=\gamma$. In this case there is nothing more to show. On the other hand, if $\gamma_0<\gamma^+$ there are only two possibilities:

1) $\chi_{.t}(0,\gamma^+)>0$, hence $\Gamma^+=\gamma^+$. Then for all $\gamma\in[\gamma_0,\gamma^+[$ one has $\chi_{.t}(0,\gamma)>0$ and $l^+(\gamma)=\log\chi(0,\gamma)>0$.

2) $\chi_{.t}(0,\gamma^+)<0$, hence $\Gamma^+=\gamma_0$. Then for all $\gamma\in]\gamma_0,\gamma^+]$ one has $\chi_{.t}(0,\gamma)<0$ and $l^+(\gamma)=\log h(\gamma)<0$.

v) In order to deal with the second term σ_n^- of $\sigma(\gamma,I_n)$ just interchange λ_i and ν_i. Then, γ^+ is replaced by γ^- and the only thing to do is to recognize, that the same γ_0 is obtained. For this just note that interchanging λ_i and ν_i and replacing t by $\gamma-t$ keeps the equations of (16) invariant. Thus

$$\lim_{n\to\infty}\frac{1}{n}\log\sigma_n^-(\gamma)\begin{cases}>0 & \text{if } \gamma<\Gamma^-,\\ <0 & \text{if } \gamma>\Gamma^-.\end{cases}$$

vi) Finally, take $\gamma<\Gamma$. Since $\sigma(\gamma,I_n)=\sigma_n^+ + \sigma_n^-$ with both terms positive, iv) and v) give $\sigma(\gamma,I_n)\ge 1$ and hence $\gamma\le\gamma_n$ for sufficiently large n. For $\gamma>\Gamma$, iv) and v) give $\sigma(\gamma,I_n)\le 1/2+1/2$ and $\gamma\ge\gamma_n$ for n large enough. Consequently, $\lim\gamma_n=\Gamma$ and the theorem follows. $\diamond$

4 APPLICATIONS

In this section theorem 4 is compared with results from [5], [3], [9] and [6].

Falconer [5] gave a lower bound for the Hausdorff dimension of self-affine sets, which does not require a particular form of the 'open set' as in (5), but which does not apply to connected invariant sets. In our context, his result reads as follows: Given a linear transformation S on $\mathbb{R}^s$ with singular values $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_s$, the singular value function ϕ^β is for positive β defined by

$$\phi^\beta(S) = \begin{cases} \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{m-1} \cdot \alpha_m^{\beta+1-m} & \text{if } \beta \leq s, \\ (\alpha_1 \cdot \ldots \cdot \alpha_s)^{\beta/s} & \text{otherwise}, \end{cases}$$

where $m = \lceil \beta \rceil$. For a family $w_i(x,y) = S_i(x,y) + (u_i, v_i)$ $(i = 1, \ldots, r)$ of affine transformations with the OSC (3) in the plane denote by $d_-(w_1, \ldots, w_r)$ the unique β satisfying

$$\lim_{n\to\infty} \Big(\sum_{\underline{i}\in I_n} (\phi^\beta(S_{\underline{i}}^{-1}))^{-1} \Big)^{1/n} = 1.$$

Theorem 5 (Falconer) *Let K be the invariant set of $w_1, \ldots, w_r$ as above. If the sets $w_i(K)$ are mutually disjoint, then $d_{\mathrm{HD}}(K) \geq d_-(w_1, \ldots, w_r)$.*

In the case of diagonal affinities (1), the bound $d_-(w_1, \ldots, w_r)$ can be calculated using similar methods as in the previous section. Let

$$\varphi_{\underline{i}}(\beta) = \begin{cases} \lambda_{\underline{i}}^\beta & \text{if } \beta \leq d', \\ \lambda_{\underline{i}} \nu_{\underline{i}}^{\beta-1} & \text{if } d' < \beta, \end{cases} \quad \text{and} \quad \theta_{\underline{i}}(\beta) = \begin{cases} \nu_{\underline{i}}^\beta & \text{if } \beta \leq d'', \\ \nu_{\underline{i}} \lambda_{\underline{i}}^{\beta-1} & \text{if } d'' < \beta. \end{cases}$$

Then

$$\tau_n(\beta) := \sum_{\underline{i}\in I_n} \Big(\phi^\beta(S_{\underline{i}}^{-1})\Big)^{-1} = \sum_{\underline{i}\in I_n^+} \theta_{\underline{i}}(\beta) + \sum_{\underline{i}\in I_n^-} \varphi_{\underline{i}}(\beta), \tag{19}$$

and, following the lines of section 3let β^+ and β^- denote the unique numbers satisfying

$$\sum_{i=1}^r \theta_i(\beta^+) = 1 \quad \text{resp.} \quad \sum_{i=1}^r \varphi_i(\beta^-) = 1.$$

If (15) holds, denote the unique solution of

$$\left| \begin{array}{rcl} \sum_{i=1}^r \theta_i(\beta)(\lambda_i/\nu_i)^t & = & 1 \\ \sum_{i=1}^r \log(\lambda_i/\nu_i)\theta_i(\beta)(\lambda_i/\nu_i)^t & = & 0 \end{array} \right|$$

by (t_0, β_0). Otherwise set $\beta_0 = 0$. Finally let

$$B^+ := \begin{cases} \beta^+ & \text{if } \sum_{i=1}^r \log(\lambda_i/\nu_i)\theta_i(\beta^+) \geq 0 \\ \beta_0 & \text{otherwise} \end{cases} \quad B^- := \begin{cases} \beta^- & \text{if } \sum_{i=1}^r \log(\nu_i/\lambda_i)\varphi_i(\beta^-) \geq 0 \\ \beta_0 & \text{otherwise} \end{cases}$$

Proposition 6 *For diagonal affine contractions, $d_-(w_1, \ldots, w_r) = B := \max(B^+, B^-)$.*

Proof Taking care to the special values $\beta = d'$ and $\beta = d''$, where φ resp. θ are not differentiable in general, the proof of theorem 4 carries over posing no essential problems. $\diamond$

A comparison of B and Γ is easy. Since d_- may not only involve the smaller of the two singular values of w_i but also the larger ones, one has $\tau_n(\gamma) \geq \sigma(\gamma, I_n)$ by (11) and (19), and hence $B \geq \Gamma$. On the other hand, if $d'' \leq d'$, $\sum \lambda_i^{d''} \leq 1$ and $\sum \nu_i^{d''} \leq 1$ as in Ex. 2, one has $B = \Gamma \leq d''$ due to $\tau_n(\gamma) = \sigma(\gamma, I_n)$ ($\gamma \leq d''$). Moreover, $\beta_0 = \gamma_0$ holds always, allowing further situations with $B = \Gamma$ (see Ex. 3). Though Γ can never exceed B, it is useful, since it applies also to connected sets.

Falconer also provided an 'almost sure' value d_F of $d_{\mathrm{HD}}(K)$ of self-affine sets [3]. Using similar methods as in section 3, [9] was able to give an explicit formula for d_F. This reads as:

Let K be the compact set invariant under some family of diagonal affine contractions $w_1, \ldots, w_r$ of $\mathbb{R}^d$. Provided $\lambda < 1/3$

$$d_{\mathrm{HD}}(K) = d_{\mathrm{box}}(K) = d_F := \max(\beta^-, \beta^+)$$

for almost every choice of $(u_1, v_1, \ldots, u_r, v_r)$ with respect to Lebesgue measure in $\mathbb{R}^{dr}$.

Note, that the OSC is not required. Moreover, $d_{\mathrm{box}}(K) \leq d_F$ for all $(u_1, v_1, \ldots, u_r, v_r)$. For the actual value of $d_{\mathrm{box}}(K)$, which is well known to be an upper bound of $d_{\mathrm{HD}}(K)$, we refer again to [9], where the generalized dimensions D_q and the multifractal spectrum of self-affine measures ($\mu = \sum p_i \cdot \mu(w_i^{-1}(\cdot))$) are calculated. It is worth noting, that the spectrum of these measures show features, which can not be observed in the self-similar case: the function $q \mapsto D_q$ may be not differentiable or once but not twice differentiable. However, of interest here is the special value D_0 which equals the box dimension of the support K of μ.

Let K be a diagonal self-affine set. Assume that $D^{(k)} = d_{\mathrm{box}}(K^{(k)})$ exist for $k = 1, 2$, where $K^{(1)}$ and $K^{(2)}$ are the projections of K onto the invariant subspaces $\mathbb{R}^{d'}$ and $\mathbb{R}^{d''}$ respectively. Then

$$d_{\mathrm{box}}(K) = \max(d^+, d^-), \tag{20}$$

where d^+ and d^- are defined through

$$\sum_{i=1}^{r} {\lambda_i}^{D^{(1)}} {\nu_i}^{(d^+ - D^{(1)})} = 1 \quad \textit{resp.} \quad \sum_{i=1}^{r} {\nu_i}^{D^{(2)}} {\lambda_i}^{(d^- - D^{(2)})} = 1.$$

Remark Provided $\lambda_i \geq \nu_i$ $(i = 1, \ldots, r)$, one has $\Gamma = \gamma^+$, $B = \beta^+$, $d_F = \beta^-$ and $d_{\mathrm{box}}(K) = d^+$. In the case of self-similar sets ($\lambda_i = \nu_i$, $i = 1, \ldots, r$), all values coincide.

Example 1 (Gatzouras, Lalley) In [6] certain special cases of diagonal self-affine sets K with $\lambda_i \leq \nu_i$, called 'carpets' [8], have been investigated. In particular, the Hausdorff dimension of carpets is shown to satisfy a variational principle which involves the invariant measures supported on K (see [10]). Moreover, $d_{\mathrm{HD}}(K) = d_{\mathrm{box}}(K) = \delta$ iff $0 < m^\delta(K) < \infty$. Both results are of great interest. However, explicit calculation of $d_{\mathrm{HD}}(K)$ seems hopeless in general and bounds such as $\Gamma = \gamma^-$ and $B = \beta^-$ may be useful. ○

Example 2 Consider the maps $w_i(x, y) = (x/4, y/8) + (u_i, v_i)$ $(i = 0, \ldots, 3)$ with the round set $]0, 1[^2$. Since $\lambda_i \geq \nu_i$ for all i, one finds $\Gamma = B = \gamma^+ = 2/3$, $d_F = \beta^- = 1$. Hence, $d_{\mathrm{HD}}(K) = d_{\mathrm{box}}(K) = 1$ for almost all (u_i, v_i) with respect to Lebesgue measure in the $\mathbb{R}^8$, $2/3 \leq d_{\mathrm{HD}}(K)$ for all (u_i, v_i) which imply the round OSC and $d_{\mathrm{box}}(K) \leq 1$ for all choices of (u_i, v_i). Finally, $d_{\mathrm{box}}(K) = d^+$, which depends analytically on $D^{(1)}$.

For $u_i = 0$ and $v_i = i/4$, K is a self-similar set lying on the y-axis with $d_{\mathrm{HD}}(K) = d_{\mathrm{box}}(K) = \log 4/\log 8 = 2/3 = \Gamma$. For $a_0 = a_2 = 0$, $a_1 = a_3 = 3/4$, $b_0 = b_1 = 0$ and $b_2 = b_3 = 7/8$, one obtains the product of two self-similar sets with $d_{\mathrm{HD}}(K) = d_{\mathrm{box}}(K) = 1/2 + 1/3 = 5/6$ ([11], [8] or [6]). Finally, for $u_i = i/4$ and $v_i = 0$ one finds $d_{\mathrm{HD}}(K) = d_{\mathrm{box}}(K) = 1$. ◯

Example 3 (Rosette) Consider the maps

$$\begin{aligned} w_1(x,y) &= (x/2 - 1/2, y/4) \qquad & w_2(x,y) &= (x/2, y/2 - 1/2) \\ w_3(x,y) &= (x/2 + 1/2, y/4) \qquad & w_4(x,y) &= (x/2, y/2 + 1/2) \end{aligned}$$

with the round open set $O = \{(x,y) : |x| + |y| < 1\}$ (Fig. 2). Here, $D^{(1)} = D^{(2)} = 1$, and

$$\Gamma = \gamma_0 = 4/3 \le d_{\mathrm{HD}}(K) \le d_{\mathrm{box}}(K) = 3 - \frac{\log(\sqrt{17} - 1)}{\log 2} \simeq 1.357,$$

a satisfying bound. Also $d_-(w_1, \ldots, w_r) = \beta_0 = 4/3$, but the sets $w_i(K)$ are not disjoint. ◯

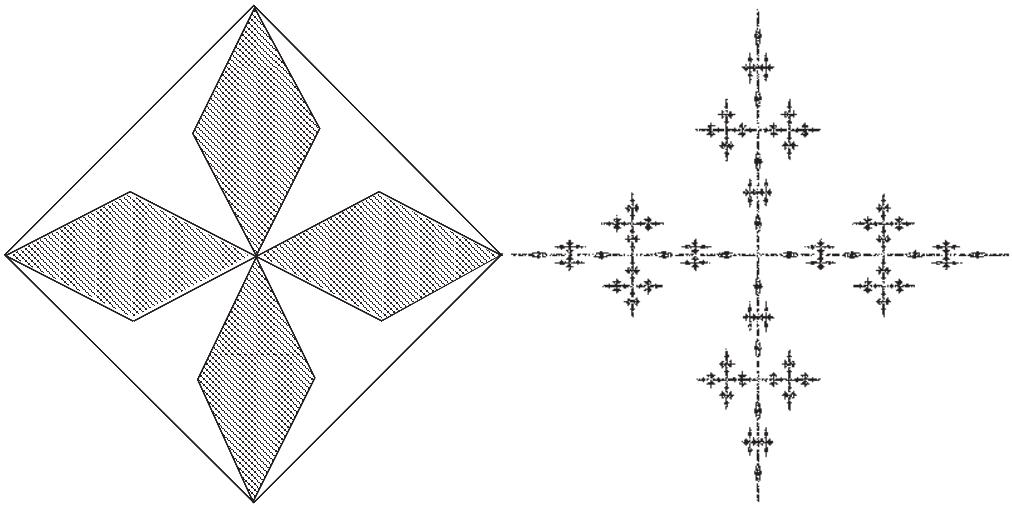

Figure 2: The construction of the rosette (see Ex. 3).

5 CONCLUSIONS

We presented a class of self-affine sets and measures which is wide enough to cover important applications such as fractal interpolation surfaces and mountain surfaces. On the other hand, the affine transformations used are simple enough to allow the explicit calculation of various fractal characteristics such as bounds for the Hausdorff dimension, the box dimension and the multifractal spectrum. As we stressed, with self-affine sets and measures one may not always get the intuitive answer: The spectrum does not have to be smooth. Furthermore, although the dimension of self-affine sets is 'almost surely' known, exceptions do occur. Our explicit bounds give an idea, to what extent the effective value may differ from the expected one.

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