Multicast Capacity of Large Homogeneous Multihop Wireless Networks

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***Abstract*—Most existing work on multicast capacity of large homogeneous networks is based on a simple model for wireless channel, namely the Protocol Model [12], [19], [22]. In this paper, we exploit a local capacity tool called *arena* which we introduced recently in order to render multicast accessible to analysis also under more realistic, and notably less pessimistic channel models. Through the present study we find three regimes of the multicast capacity (λ_{m}) for a homogeneous network depending on the ratio of terminals among the nodes of the network. We note that the upper bounds we establish under the more realistic channel assumptions are only $\sqrt{\log(n)}$ larger than the existing bounds. Further, we propose a multicast routing and time scheduling scheme to achieve the computed asymptotic bound over all channel models except the simple Protocol Model. To this end, we employ percolation theory among other analytical tools.**

Finally, we compute the multicast capacity of large mobile wireless networks. Comparing the result to the static case reveals that mobility increases the multicast capacity. However, the mobility gain decreases when increasing the number of terminals in a fixed size mobile network.

I. Introduction

There has been a growing interest to understand the fundamental capacity limits of wireless networks [5], [8], [9], [13], [14], [18], [19], [25]. Results on network capacity are not only important from a theoretical point of view but may also inform protocol design and architecture of wireless networks. In this paper, we study the capacity of wireless networks for *multicast* applications. We focus on *large homogeneous wireless networks* which is a popular model in network capacity papers. To set notation, we assume n nodes ($n \to \infty$) in the network and multiple multicast sessions where each session has a source and $k-1$ terminals. Also, we assume *random traffic* among the nodes, that means the source and terminals of each multicast session are selected uniformly random among the nodes. For the above topology and traffic pattern, we define the *multicast capacity* (λ_{m}) as the maximum rate of generation of multicast bits such that all bits can be delivered to their terminals successfully in a limited time, almost surely (a.s.) as $n \to \infty$.

Multicast capacity of planar large homogeneous networks has been studied in several papers [12], [19], [22]. Very recently, [19] computes asymptotic bounds on the multicast capacity showing that $\lambda_{\mathrm{m}} = \Theta\left(W\sqrt{n/k\log(n)}\right)$ when $k = O(n/\log(n))$, and $\lambda_{\mathrm{m}} = \Theta(W)$ when $k = \Omega(n/\log(n))$, where W is the wireless channel capacity; note that the latter is asymptotically equal to the broadcast capacity [13]. Further, [22] studies the multicast capacity for the case where the number of sources is n^{ϵ} $(0 < \epsilon < 1)$ and the number of terminals is $n^{1-\epsilon}$ for each multicast session, then it proves that $\lambda_{\mathrm{m}} = \Theta\left(W\sqrt{n^{\epsilon}/\log(n)}\right)$. We point out that these papers have used the simple Protocol Model (see Section III) for modeling the wireless channel in their analysis. In this paper, we study the multicast capacity using other channel models such as the Physical Models; these more advanced models are considered to be more realistic and accurate.

We prove that we have $\lambda_{\mathrm{m}} = \Theta(W\sqrt{n/k})$ when $k = O(n/\log(n)^3)$ for all models found in the literature, with one exception: for the simple Protocol Model. The reason that we find different bounds here is that the simple Protocol Model makes pessimistic assumptions on interference when compared to the other channel models. A similar situation has been found for the unicast capacity and transport capacity [5], [14].

Furthermore, we prove that for all channel models, when $k = \Omega(n/\log(n))$ then $\lambda_{\mathrm{m}} = \Theta(W)$. This is a generalization to the result of previous work [19] that was proved based on the simple Protocol Model. Note that the use of novel techniques were required to achieve this extension. Particularly, we use *local capacity* techniques which we introduced very recently in [14] to provide upper bounds on the multicast capacity; these techniques are useful in other contexts too.

As the first and the second contributions of this paper, we provide asymptotic bounds on the multicast capacity. First, we find two novel upper bounds on the multicast capacity. These bounds are different than the computed bounds in the recent papers [12], [19], [22]. Combining these bounds provides a tight upper bound on the multicast capacity with three asymptotic regimes in terms of k and n. Second, we introduce some multicast routing schemes and MAC layer time scheduling for achieving a multicast throughput within a constant factor of the computed upper bounds. Interestingly, we apply some results from *percolation theory* and *connectivity of large homogeneous networks* to prove the achievability in different asymptotic regimes.

As the third contribution, we study the multicast capacity of large mobile wireless networks. We prove that multicast capacity is $\Theta(Wn/k)$; this implies that mobility can increase the multicast capacity by at least factor of $\Omega(\sqrt{n/k})$. Also, it demonstrates that the mobility gain reduces by increasing the number of terminals in a fixed mobile network. In particular case where $k = n$ which represents broadcast, mobility does not change the capacity asymptotically.

The paper is organized as follows. In Section II, we summarize related work on the network capacity. We introduce the network models and basic notations in Section III. In Section IV and Section V, we compute novel upper bounds and lower bounds for the multicast capacity. We compute multicast capacity of large mobile networks in Section VI. Finally, we conclude the paper in Section VII.

II. Related Work

Gupta and Kumar [9] study the network capacity for unicast connections in static wireless networks consisting of n nodes distributed in a circle of area A. They define the "transport capacity" of a wireless network with units of bit-meters per second as the maximum rate of the packets times the distance they travel between the source and the destination. Their main result says that the aggregate transport capacity of unicast connections is $O(W\sqrt{An})$ in an arbitrary network and it is $\Theta(W\sqrt{An/\log(n)})$ in a large homogeneous network. As a result, if the capacity is shared between random sources and destinations in the network, per node capacity decreases as $O(W\sqrt{1/n})$ (in homogeneous network $\Theta(W\sqrt{1/n\log(n)})$) when n grows. The same authors also analyze three dimensional networks [10]. They prove that if the nodes are distributed in a sphere with volume V then the aggregate transport capacity is $O(W\sqrt[3]{Vn^2})$ [10]. Later, these results were generalized for a more accurate channel model in [1].

Franceschetti *et al.* [5] propose a constructive technique for large homogeneous networks which achieves a per-node capacity of $\Theta(W\sqrt{1/n})$. This is an improvement of factor $\sqrt{\log(n)}$ over earlier result of $\Theta(W\sqrt{1/n\log(n)})$ [9], [17]. The main idea of the technique is build backbone of nodes that carry packets across the network at constant rate, using short-hops transmissions (size of $\simeq \sqrt{A/n}$), and to drain the rest of the traffic to the backbone using a single long-hop transmission (size of $\simeq \sqrt{A\log(n)/n}$). They proved the existence of such a backbone is due to percolation theory [7].

For wireless mobile networks, Grossglauser and Tse [8] show that per node capacity can be increased to $\Theta(W)$ if packet delay is left unbounded. They propose a mobility-based routing method in which the number of retransmissions of the unicast packets between source and destination is reduced to two. They consider a mobility model where the nodes move uniformly and independently within a circular area; a mobile node close to the source receives the packet and moves in the entire network randomly and later delivers the packet when it is close enough to the destination. Many other efforts demonstrate that there is a trade-off between the capacity and the delay in wireless mobile networks, for different mobility patterns and constraints on delay (see [23] for references).

Introducing a new direction in network capacity research, in our pervious work, we present *local-capacity* concept and find new capacity bounds based topology and traffic pattern for unicast and multicast applications in arbitrary wireless networks under all classical channel models [14]. We employed a new analytical tool called *arena-rate function* to bound the capacity of wireless networks. In this paper we apply this tool and some of analytical results of to [14] to upper bound the multicast capacity.

It should also be mentioned that there exists work on the *broadcast capacity* of wireless networks [13], [15], [24], [27]. The recent papers prove that the broadcast capacity is $\Theta(W)$ in "well-connected" arbitrary wireless networks [13], [15]. More recently, some papers compute the multicast capacity and unify the existing results on unicast and broadcast capacity [19], [22]. We discussed about the results of there papers and the contribution of our present work in the introduction section.

Note that all the above mentioned papers as well as this paper assume only point-to-point coding at the receivers. If the nodes are allowed to use sophisticated multi-user coding then network capacity of a higher order than that described above can be achieved [6], [11], [16], [20]. A full discussion of these results is beyond the scope of this paper.

III. Wireless Channel Models and Basic Notions

In this section, we describe the models and notions used in this paper. We consider a wireless network consisting of n wireless nodes in d-dimensional space cube with volume V.

We denote the set of transmitter-receiver pairs of *simultaneous direct* transmissions active at time τ by $\mathcal{SD} := \{(S_1, D_1), (S_2, D_2), \ldots, (S_u, D_u)\}$. Also, we denote the set of transmitters by $\mathcal{S} := \{S_1, \ldots, S_u\}$. Note that these sets vary over time; if not otherwise indicated, however, we will consider one fixed but arbitrary time instant. For simplicity in notation, the node symbols are used also to represent their locations. For example, $|S_i - D_i|$ is the Euclidean distance between the nodes S_i and D_i in $\mathbb{R}^d$.

A. Wireless Channel Models

Here, we briefly review of the common channel models found in the literature on wireless network capacity. These models have been explained with more details in [14]. First, the *Protocol Model* models a successful transmission based on the distance with the closest interfering transmitter. This model is the simplest of the three and easiest to analyze. Second, the *Physical Model* sets a threshold on the *Signal to Interference plus Noise Ratio* (SINR) of the received signal, declaring the transmission to be successful if the SINR is larger than the threshold. Third, the *Generalized Physical Model* determines the transmission rate in terms of the SINR by using Shannon's capacity formula for a wireless channel with additive Gaussian white noise [3].

1) Protocol and Physical Model: In both, the Protocol and the Physical Model the assigned transmission rate from node $S_i \in \mathcal{S}$ to node D_i is $W_i = W$ if the transmission is modeled as successful, and the rate is zero for unsuccessful transmissions.

These models are different on the conditions for successful transmission. In the literature [9], [10] one finds the following three different versions under the term "Protocol Model". Given the *interference parameter* $\Delta > 0$ a transmission is modeled as successful if:

- Protocol Model 1:
  $|S_j - D_i| \geq (1+\Delta)|S_j - D_j|$ for all $S_j \in \mathcal{S}\backslash\{S_i\}$.
- Protocol Model 2:
  $|S_j - D_i| \geq (1+\Delta)|S_i - D_i|$ for all $S_j \in \mathcal{S}\backslash\{S_i\}$.
- Protocol Model 3 or "simple Protocol Model":
  $|S_j - D_i| \geq (1+\Delta)r$ for all $S_j \in \mathcal{S}\backslash\{S_i\}$, and $|S_i - D_i| \leq r$ where the *transmission range* r is an additional parameter.

Under the Physical Model a transmission is modeled as successful if

$$\mathrm{SINR}_i = \frac{P_i G_{ii}}{N_o + \sum\limits_{j \neq i, j \in \mathcal{S}} P_j G_{ji}} \geq \beta \tag{1}$$

Here, β is the SINR-threshold, N_o represents the ambient noise, and G_{ji} denotes the signal loss, meaning that $P_j G_{ji}$ is the receiving power at node D_i from transmitter S_j. We assume a low power decay for the signal loss of the form $G_{ji} = |S_j - D_i|^{-\alpha}$, where $\alpha > d$ is the signal loss exponent.

2) Generalized Physical Model: In this model all node pairs are able to communicate by direct transmission, however with a rate W_i that depends on SINR as

$$W_i = W \cdot \log_2(1 + \mathrm{SINR}_i) \tag{2}$$

While this model assigns a more realistic transmission rate at large distance than the other two channel models, it also results in a singularity under the signal loss model $G_{ii} = |S_i - D_i|^{-\alpha}$: according to (2) the receiving power and the rate are amplified to unrealistic levels if transmitter and receiver are placed very closely to each other. The singularity can be easily addressed by bounding the received power at each node [2], [4]. Additionally, we assume that the maximum transmission power (P) is bounded. Therefore, we have $W_i = O(W)$. Note that limiting the maximum transmission power is an important assumption here, it has been shown that a significant transmission power (compare to the network size) can change the asymptotic bounds on network capacity [15].

B. Local Capacity Tools

Here, we review local capacity tools which have been developed in [14]. A *transmission arena* (A_i) is a shape that is defined based on location of the transmitter (S_i) and the receiver (D_i) for every transmission in the network. In this paper, we define $A_i = \{X : |S_i - X| \leq |S_i - D_i|\}$ (a circular area around the transmitter with radius of $|S_i - D_i|$) as transmission arena. It has been proved in [14] that for any arbitrary point X in the space, at any time instance τ, and for any set of simultaneous transmitters $\mathcal{S}$,

$$\sum_{S_i \in \mathcal{S}} W_i \cdot \mathbb{I}_{A_i}(X) \leq MW \tag{3}$$

where M is called *arena-bound* and $\mathbb{I}_{A_i}(X)$ is an indicator showing whether point X is located inside A_i or not. We call equation (3) the *local capacity constraint* at point X.

The value of M can be computed in terms of channel model parameters as given in Lemma 1 (the proof in [14]). Clearly, $M = \Theta(1)$ for Protocol Models and Physical Model.

Lemma 1: The arena-bounds (M) can be chosen for different channel models as

$$M = \begin{cases} 1 & \text{for } \Delta > 2\text{, any Protocol Model,} \\ \lceil \frac{(4+2\Delta)^d}{\Delta^{2d}} - 1 \rceil & \text{for } \Delta \leq 2\text{, Protocol Models 1, 2,} \\ \lceil \frac{(2+\Delta)^d}{\Delta^d} - 1 \rceil & \text{for } \Delta \leq 2\text{, Protocol Model 3,} \\ \left\lceil \frac{3^\alpha P_{\max}}{\beta P_{\min}} \right\rceil & \text{for Physical Model.} \end{cases} \tag{4}$$

Here, $P_{\max}$ and $P_{\min}$ are the maximum and minimum transmission power of the nodes.

It has been shown in [14] that $M = O(\log(n))$ for Generalized Physical Model. However, as we will explain later, for a particular case that we apply (3), $M = \Theta(1)$.

Also, [14] defines *filled volume space* for every transported bit b in the wireless network as

$$\sigma_b = \sum_{i \in \mathcal{H}_b} \int_{x \in \mathrm{cube}} A_i(X) dX \tag{5}$$

where $\mathcal{H}_b$ is the set of transmissions used for transmitting the the bit b.

Lemma 2 provides a capacity bound based on the filled volume of space for transporting the bits of a particular multicast application $\mathfrak{app}$. The proof can be found in Theorem 6 of [14].

Lemma 2: Assume that $s_{\mathfrak{app}} \leq (\sum_{b_j} \sigma_{b_j})/(\sum_{b_j} 1)$ as $T \to \infty$ where $\{b_k\}_k$ are the transported bits under application $\mathfrak{app}$ in time interval $[0, T]$. Then the rate of generation of successfully transported bits of the application ($\lambda_{\mathfrak{app}}$) is bounded as

$$\lambda_{\mathfrak{app}} \leq MW \cdot V / s_{\mathfrak{app}} \tag{6}$$

C. Transport Capacity

The transport capacity is useful to analyze network capacity for a given set of unicast source-destination pairs $\mathcal{AB} := \{(A_1, B_1), ..., (A_u, B_u)\}$. It can be defined as [9]:

$$C_T(\mathcal{AB}) := \max_{\text{multi-hop paths}} \sum_j |A_j - B_j| R_j \tag{7}$$

where R_j is the average rate of unicast connection between of A_j and B_j over a given multi-hop path. The maximum is taken over all possible multi-hop routes establishing the required connections between the sources and destinations. The bounds on transport capacity are presented in [1], [9], [10], [14].

IV. Upper Bounds on the Multicast Capacity

In this section, we find novel upper bounds on the multicast capacity. We drive two theorems. The first theorem takes into account the *homogeneity* of network topology for computing the upper bound. The second theorem uses the *randomness* property of the topology to provide a new capacity bound.

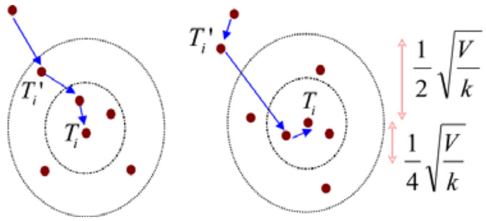

Fig. 1. Grouping the terminals of a multicast session by considering the last node (T_i') which transmits the packet inside the small ball around a terminal (T_i). In the left figure $T_i \in \mathcal{I}_1^{(u)}$ and in the right figure $T_i \in \mathcal{I}_2^{(u)}$.

A. Upper-bound based on Homogeneity of the Topology

Here, we explain how to upper bound the multicast capacity when the network has some homogeneity on the distribution of nodes in the space. Consequently, the results of this subsection are applied also for wireless network with grid topology in d-dimensional space.

Theorem 1: Assume a homogeneous wireless network in d-dimensional space. Then, for all channel models

$$\lambda_{\mathrm{m}} = O\left(\sqrt[d]{(\frac{n}{k})^{d-1}}W\right) \tag{8}$$

almost surely for large n.

Proof of Theorem 1: Without lack of generality, assume that $k \to \infty$ as $n \to \infty$. Consider an arbitrary multicast session u with set of terminals $\mathcal{T}^{(u)}$ in the network. We define $\mathcal{I}^{(u)} \subset \mathcal{T}^{(u)}$ as the set of terminals such that they are apart from the rest of terminals by distance of $\sqrt[d]{\frac{V}{k}}$.

$$\mathcal{I}^{(u)}= \left\{T_i \in \mathcal{T}^{(u)}\text{s.t.}\forall T_j \in \mathcal{T}^{(u)}\setminus \{T_i\}\text{: } |T_j - T_i| \geq \sqrt[d]{\frac{V}{k}}\right\}$$

Since the set of terminals for the multicast session u have been chosen uniformly random, using Lemma 2 of [15], we can show that $\#\mathcal{I}^{(u)} \geq c_1 \cdot k$ with high probability (w.h.p.), where $c_1 = \exp(-\pi_d)/2$ is a constant and π_d is the volume of unit d dimensional sphere. This property results from the *homogeneity* of the topology.

Next, we draw two balls around each node $T_i \in \mathcal{I}^{(u)}$ with radiuses of $\frac{1}{4}\sqrt[d]{\frac{V}{k}}$ and $\frac{1}{2}\sqrt[d]{\frac{V}{k}}$. Also, for every node $T_i \in \mathcal{I}^{(u)}$, we consider the last node (T_i') that forwards the multicast packet inside the small ball (with radius of $\frac{1}{4}\sqrt[d]{\frac{V}{k}}$) around T_i. Then, we partition $\mathcal{I}^{(u)}$ into two subsets:

$$\mathcal{I}_1^{(u)} = \left\{T_i \in \mathcal{I}^{(u)}\text{: } |T_i - T_i'| \leq \frac{1}{2}\sqrt[d]{\frac{V}{k}}\right\} \tag{9}$$

$$\mathcal{I}_2^{(u)} = \mathcal{I}^{(u)} \setminus \mathcal{I}_1^{(u)} \tag{10}$$

Now, we take average (i.e. expectation) over the size of these sets for the transported multicast bits in a long period of time in different multicast sessions. We analyze these two cases separately: (i) if $\mathbb{E}[\#\mathcal{I}_1^{(u)}] \geq \mathbb{E}[\#\mathcal{I}^{(u)}]/2$ (ii) if $\mathbb{E}[\#\mathcal{I}_2^{(u)}] \geq \mathbb{E}[\#\mathcal{I}^{(u)}]/2$.

Case (i): For every multicast session u, we consider the set of transmissions which transport the packet from T_i' to T_i for all $T_i \in \mathcal{I}_1^{(u)}$. From the definition of $\mathcal{I}^{(u)}$, the large balls (with radius of $\frac{1}{2}\sqrt[d]{\frac{V}{k}}$) are disjoint in the space; hence, the transmissions are distinct for every $T_i \in \mathcal{I}_1^{(u)}$. These transmissions transport the packet for distance of at least $|T_i' - T_i| \geq \frac{1}{4}\sqrt[d]{\frac{V}{k}}$ inside each large ball. Now, we apply the existing bounds on the transport capacity of wireless networks for the described transmissions [1], [9], [10], [14]. Then,

$$\lambda_{\mathrm{m}} \cdot \mathbb{E}[\#\mathcal{I}_1^{(u)}] \cdot \frac{1}{4}\sqrt[d]{\frac{V}{k}} \leq c_2 W \sqrt[d]{Vn^{d-1}} \tag{11}$$

where c_2 is a constant given in the literature. Thus,

$$\begin{aligned}
\lambda_{\mathrm{m}} &\leq c_2 W \sqrt[d]{Vn^{d-1}}/(\mathbb{E}[\#\mathcal{I}_1^{(u)}]\frac{1}{4}\sqrt[d]{\frac{V}{k}}) \\
&\leq 4c_2 W \sqrt[d]{kn^{d-1}}/(\mathbb{E}[\#\mathcal{I}^{(u)}]/2) \\
&\overset{\text{a.s.}}{\leq} 4c_2 W \sqrt[d]{kn^{d-1}}/(c_1 k/2) \\
&= \frac{8c_2}{c_1} W \sqrt[d]{(\frac{n}{k})^{d-1}}
\end{aligned}$$

Case (ii): For every multicast session u, we consider the set of transmissions by T_i' for all $T_i \in \mathcal{I}_2$. A simple geometric computation shows that at least $\pi_d' \frac{1}{2^d}\frac{V}{k}$ of the large ball around T_i is covered by the transmission of T_i' where π_d' is a constant (e.g. $\pi_d' = \pi/12$ in 2-dimensional space). Since, the large balls are disjoint, it follows that the "filled volume of space" that is used for transporting the bits of session u is at least $\pi_d' \frac{1}{2^d}\frac{V}{k} \cdot \#\mathcal{I}_2^{(u)}$. Now, we apply Lemma 2 for the transported multicast packets. Then,

$$\lambda_m \cdot \mathbb{E}[\#\mathcal{I}_2^{(u)}] \cdot \pi_d' \frac{1}{2^d}\frac{V}{k} \leq MW \cdot V \tag{12}$$

Thus,

$$\begin{aligned}
\lambda_{\mathrm{m}} &\leq MWV/(\mathbb{E}[\#\mathcal{I}_2^{(u)}]\pi_d' \frac{1}{2^d}\frac{V}{k}) \\
&\leq \frac{2^d}{\pi_d'} MWk/(\mathbb{E}[\#\mathcal{I}^{(u)}]/2) \\
&\overset{\text{a.s.}}{\leq} \frac{2^d}{\pi_d'} MWk/(c_1 k/2) \\
&= \frac{2^{d+1}}{c_1 \pi_d'} MW
\end{aligned}$$

As we explained in Section III-B, $M = O(1)$ for all three Protocol Models and Physical Model.

Since, one of two above cases has to occur, we conclude that $\lambda_{\mathrm{m}} = O\left(W\sqrt[d]{(\frac{n}{k})^{d-1}}\right)$ ■

Note that for the Generalized Physical Model $M = O(\log(n))$. Therefore, if we follow the proof of Theorem 1, then gives $\lambda_{\mathrm{m}} = O(\max\{W\sqrt[d]{(\frac{n}{k})^{d-1}}, W\log(n)\})$. Anyway, this does not change the final upper bound that we provide at the end of this section in (17).

B. Upper-bound based on Randomness in the Topology

Here, we compute a new upper bound on the multicast capacity. Theorem 2 uses the randomness of the location of the nodes in order to bound the multicast capacity. Randomness of topology creates some clusters of nodes which are relatively isolated from the rest of the nodes. These clusters can at as bottleneck on the multicast capacity.

Theorem 2: Assume a homogeneous wireless network in d-dimensional space. Then, for all channel models

$$\lambda_{\mathrm{m}} = \begin{cases} O\left(\frac{n}{k\log(n)}W\right) & \text{if } k \leq \frac{n}{\log(n)} \\ O(W) & \text{if } k \geq \frac{n}{\log(n)} \end{cases} \tag{13}$$

almost surely for large n.

Proof of Theorem 2: The idea of the proof is to show that there exists a cluster of nodes in a random topology which is relatively isolated from the rest of the nodes and the average rate of information that can be sent/received by the nodes of this cluster is very limited compare to size of cluster. This method gives us a new bound on the multicast capacity.

For the proof, we divide V into cube cells with side size of $\frac{1}{3}\sqrt[d]{\frac{V\log(n)}{n}}$. Now, consider a fixed arbitrary cell. Denote the number of nodes in the cell by η. Let $p_1 = \mathbb{P}[\eta = 0]$, then as $n \to \infty$,

$$\begin{aligned} p_1 &= \left(1 - \frac{1}{3^d}\frac{V\log(n)}{n}/V\right)^n \\ &= \left(1 - \frac{3^{-d}\log(n)}{n}\right)^n \\ &\simeq e^{-3^{-d}\log(n)} \\ &= n^{-3^{-d}} \end{aligned}$$

Also, let $p_2 = \mathbb{P}[\eta < (1-\delta)3^{-d}\log(n)]$ where $0 < \delta < 1$ is an arbitrary constant number. We can find an upper bound on p_2 by applying the Chernoff's bound.

$$p_2 \leq e^{-\delta^2 3^{-d}\log(n)/2} = n^{-\delta^2 3^{-d}/2} \tag{14}$$

Next, we analyze event B which is the event that there exists a group of 3^d neighbor cells, such that the middle cell contains at least $(1-\delta)\log(n)$ nodes and the $3^d - 1$ cells around it are empty.

$$\begin{aligned} \mathbb{P}[B] &= 1 - \left(1 - (1-p_2)p_1^{3^d-1}\right)^{\frac{n}{\log(n)}} \\ &\geq 1 - \left(1 - (1 - n^{-\delta^2 3^{-d}/2})n^{-3^{-d}(3^d-1)}\right)^{\frac{n}{\log(n)}} \\ &\simeq 1 - \left(1 - n^{-1+3^{-d}}\right)^{\frac{n}{\log(n)}} \\ &\simeq 1 \end{aligned}$$

So, such 3^d neighbor cells in the network can be found w.h.p.. Note that we can even make a stronger argument and using Borel-Cantelly lemma [21] prove that event B occurs almost surely for large n.

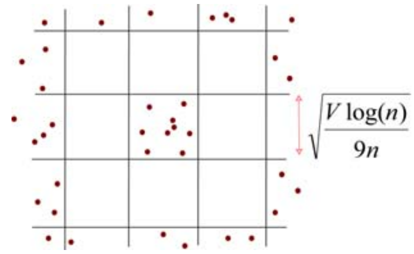

Fig. 2. A cluster of nodes in a large homogeneous network which is relatively isolated from the rest of the nodes.

Let p_3 be the probability that at least one node in the middle cell is source or terminal of a multicast session. It is easy to show that

$$1 - \left(1 - \frac{(1-\delta)3^{-d}\log(n)}{n}\right)^k \leq p_3 \leq 1 - \left(1 - \frac{e3^{-d}\log(n)}{n}\right)^k \tag{15}$$

where these bounds are obtained using $(1-\delta)3^{-d}\log(n) \leq \eta \leq 3^{-d}e\log(n)$ w.h.p.. The latter upper bound on η has been proved in Claim 3.1 of [17].

We note that in (15) if $k = O(\frac{n}{\log(n)})$ then $p_3 = \Theta(\frac{\log(n)k}{n})$ and if $k = \Omega(\frac{n}{\log(n)})$ then $p_3 = \Theta(1)$.

Now, we consider $3^d - 1$ points which are the centers of the empty cells. When a transmission between the nodes of middle cell and the rest of the nodes occurs, at least 1 of these points are contained inside the arena of that transmission. Next, we apply the local capacity constraint (3) for these $3^d - 1$ points to bound input/ouput rate of the cluster, then we find

$$\lambda_{\mathrm{m}} \cdot p_3 \leq (3^d - 1) \cdot MW \tag{16}$$

This completes the proof when $M = \Theta(1)$ which the case for Protocol and Physical models. For Generalized Physical Model we prove in the appendix that $M = \Theta(1)$ for the particular case that is studied here. ■

C. Upper-bound Regimes for Multicast Capacity

Here we combine the results of last two theorems. We find the following upper bound on the multicast capacity for all channel models. Note that for $d \geq 2$ dimensional space, multicast capacity has three regimes in terms of k. For $d = 1$ we simply have multicast capacity equal to broadcast capacity asymptotically ($\Theta(W)$).

$$\lambda_{\mathrm{m}} = \begin{cases} O(\sqrt[d]{(\frac{n}{k})^{d-1}}W) & \text{if } k \leq \frac{n}{\log(n)^d} \\ O(\frac{n}{k\log(n)}W) & \text{if } \frac{n}{\log(n)^d} \leq k \leq \frac{n}{\log(n)} \\ O(W) & \text{if } k \geq \frac{n}{\log(n)} \end{cases} \tag{17}$$

Fig. 3 shows the upper bound of multicast capacity in $d = 2$ dimensional space. Note that the previous result on multicast capacity which has been computed for Protocol Model 3 is smaller than our computed bound for $k = O(\frac{n}{\log(n)})$. The difference results from channel modeling. Protocol Model 3 which somehow models the interference of wireless channel pessimistically in comparison to other channel models. The same issue has been found for unicast capacity and transport capacity cases in [5], [14].

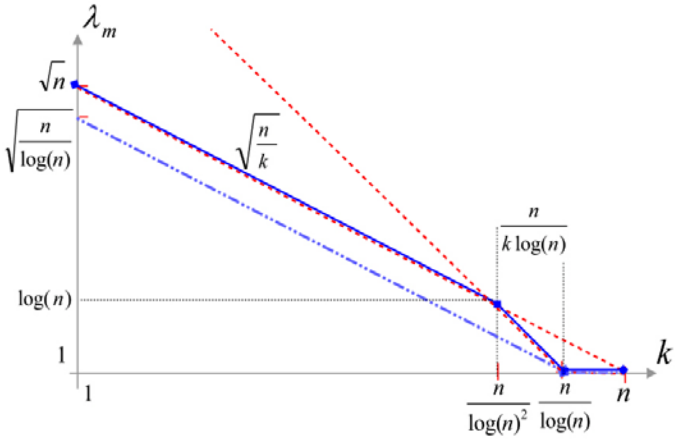

Fig. 3. Upper bound on multicast capacity in terms of k. The lower line represents multicast capacity for Protocol Model 3 and upper line shows the capacity bound for other channel models.

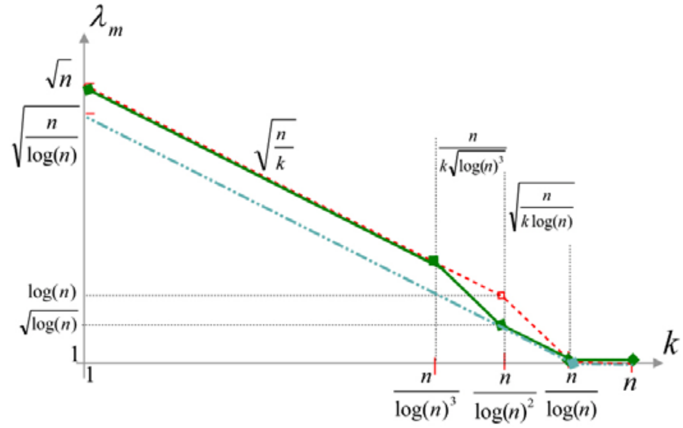

Fig. 4. Lower bound on multicast capacity in terms of k. The lower line represents multicast capacity for Protocol Model 3 and upper line shows the capacity bound for other channel models.

V. Lower Bounds on the Multicast Capacity

In this section, we introduce multicast routing and time scheduling schemes to achieve the computed upper bounds in the last section and also to provide novel lower bounds on the multicast capacity. As we explained in the introduction, the multicast capacity has been analyzed under Protocol Model 3 in [19], [22]. Here, we focus on other classical channel models (see Section III-A).

We assume that the maximum transmission power is large enough for creating a proper connectivity among the nodes for the Physical Models explained in Section III-A. Otherwise, poor connectivity of some relatively isolated single nodes can result significant decrease on the multicast capacity and change the asymptotic capacity bounds. The full discussion on the relation between maximum power level and multicast capacity is beyond the scope of this paper. This problem has been studied explicitly for the broadcast capacity in [15].

Theorem 3 provides lower bound on the multicast capacity of planar homogeneous networks in 4 cases. Fig. 4 depicts the lower bound in terms of k. In case 1 and 4, the lower bound is tight i.e. it approximates the multicast capacity up to a constant factor. In case 2 and 3, the lower bound is different than the computed upper bound in (17) by at most a factor of $O(\sqrt{\log(n)})$ which occurs for $k = \Theta(\frac{n}{\log(n)^2})$.

We present Theorem 3 for $d = 2$ dimensional space. The proof can be easily extended to $d = 3$ dimensional space. Achievability of multicast capacity in $d = 1$ dimensional space is trivially proved using the proposed broadcast schemes of [13], [15].

For the proof of Theorem 3, we construct specific cellular structures and then we review some results from existing work which reveal connectivity properties of these structures. We build three cellular structures by dividing the square area V into smaller squares with side size of $c\sqrt{\frac{V}{n}}$, $c\sqrt{\frac{V\log(n)}{n}}$, $c\log(n)\sqrt{\frac{V}{n}}$, and we call them "cells", "large-cells", and "super-cells" respectively. Parameter c is a constant, here, we set $c = 10$ which is large enough to hold the connectivity properties of cellular structures needed for our routing scheme. We call two non-empty squares (containing at least one node) in these cellular structures *adjacent* if they have a common side, i.e. every internal square has 4 *neighbors*.

Next, in Lemma 3 and 4, we explain two connectivity characteristics of these structures using the existing results on *percolation theory* and *connectivity of large homogeneous networks*. The proofs of these lemmas can be found in Theorem 3 of [5] and Claim 3.1 of [17] respectively.

Lemma 3: Consider an any arbitrary $c\log(n)\sqrt{\frac{V}{n}}$ *by* $\sqrt{V}$ *rectangular area of the cells. Then, for any* $c > \log(6)$*, there exist at least* $\gamma\log(n)$ *(*γ *is a constant depending on* c *only) disjoint paths built by the adjacent non-empty cells between left side and right side of the rectangle area a.s. for large* n*.*

Now, we consider the rectangular area created by $\frac{\sqrt{n}}{c\log(n)}$ super-cells which are on the same row or on the same column. From Lemma 3 it follows that there are $\gamma\log(n)$ disjoint paths which pass through all super-cells that are located on the same row or same column. We call the disjoint paths which connect the super-cells on the same column "vertical highways" and those which connect the super-cells on the same row "horizontal highways". Note that if a vertical and a horizontal highway pass through the same super-cell then they intersect at least in one cell. Intuitively, vertical and horizontal highways connect the super-cells to each other like a grid; there are at least $\gamma\log(n)$ highways that connect every two adjacent super-cells.

We note that the distance between two nodes which are located inside two adjacent cells on a highway is less than $\sqrt{5}c\sqrt{\frac{V}{n}}$. So, we can transport the packets can on the highways by setting the transmission range is $r_s = \sqrt{5}c\sqrt{\frac{V}{n}}$. We call these transmissions "short-hop" transmissions.

Lemma 4: For any $c > \sqrt{3}$ *the number of nodes at each* *large-cell is between* 1 *and* $c^2 e \log(n)$ *a.s. for large* n.

Based on Lemma 4, every large-cell contains at least one node a.s.. Therefore, the packets can be transported through adjacent large-cells in the network. We only need to set the transmission range $r_l = \sqrt{5}c\sqrt{\frac{V\log(n)}{n}}$ for transmitting the packet between adjacent large-cells. We call the transmissions with range of r_l "long-hop" transmissions.

In Theorem 3 we apply the connectivity properties of the described cellular structures and propose routing and time scheduling schemes to provide novel lower bounds on the multicast capacity.

Theorem 3: Assume a homogeneous wireless network in 2*-dimensional space. Then for all channel models except the simple Protocol Model*

$$\lambda_{\rm m} = \begin{cases} \Omega\left(\sqrt{\frac{n}{k}}W\right) & \text{if } k \le \frac{n}{\log(n)^3} \\ \Omega\left(\frac{n}{k\sqrt{\log(n)^3}}W\right) & \text{if } \frac{n}{\log(n)^3} \le k \le \frac{n}{\log(n)^2} \\ \Omega\left(\sqrt{\frac{n}{k\log(n)}}W\right) & \text{if } \frac{n}{\log(n)^2} \le k \le \frac{n}{\log(n)} \\ \Omega\left(W\right) & \text{if } k \ge \frac{n}{\log(n)} \end{cases} \quad (18)$$

almost surely for large n.

Proof of Theorem 3: We provide multicast scheme and prove achievability for each case separately.

Case 1: We explain this case in three steps.

Step 1 (Multicast Routing): The main idea of the routing algorithm is to route a multicast packet along highways (in short-hops) from the super-cell of the source to the super-cells of the terminals. Then, drain the packet from highways toward the terminals by long-hop transmissions (see Fig. 5). The routing scheme has three phases.

Phase (i): The packet is transported horizontally in long-hops from the source to all vertical highways which pass through the super-cell of the source. Note that at most $\sqrt{\log(n)}$ long-hop transmission will be needed in this phase. We choose one of the vertical highways randomly to transport the packet for phase (ii).

Phase (ii): We use a routing scheme similar to the proposed scheme in [22] for transporting the packet on the highways from the super-cell of the source to the super-cells of the terminals.

First, the packet is transported on the chosen vertical highway in phase (i) to all super-cells which are on the same column with the super-cell of the source. Second, a random number θ between 1 and $\lfloor\sqrt{k}\rfloor$ is chosen for that packet, and the packet is transported on a horizontal highway to a row of super-cells if the row number is $q \cdot \lceil\frac{\sqrt{n}}{c\log(n)}\rceil / \lfloor\sqrt{k}\rfloor + \theta$ (the number of super-cells on the same row is $\frac{\sqrt{n}}{c\log(n)}$) where $q = 1, 2, \ldots, \lceil\sqrt{k}\rceil$. Third, the packet is transported on a vertical highway from the closest row of super-cells which haver received the packet toward the super-cell of each terminal.

Phase (iii): For the super-cell of each terminal, the packet is transposed horizontally in long-hops along adjacent large-cells from the vertical highway which carries the packet to the terminal.

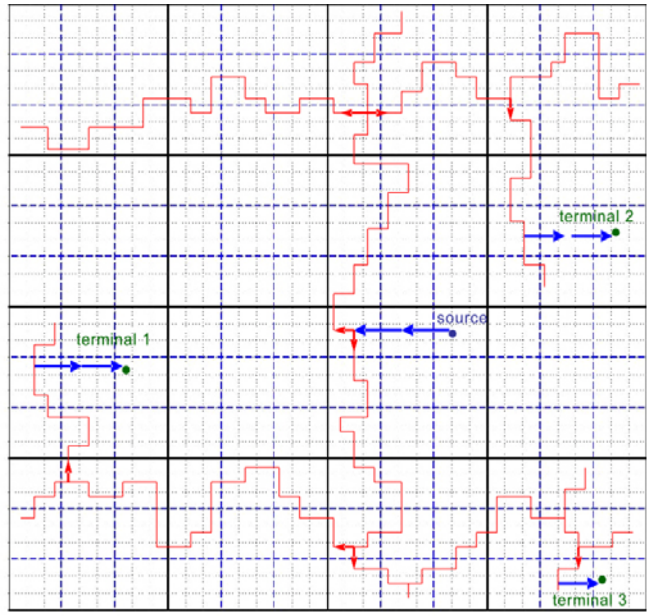

Fig. 5. The multicast packet is transported in short-hops along highways from super-cell of the source to super-cells of terminals. Then, long-hop transmissions are used to drain the packet from highway to each terminal.

Step 2 (Time-scheduling): We divide each unit time interval into two equal time slots which are used later respectively for short-hop and long-hop transmissions. At each time slot, we schedule the short-hop and long-hop transmissions based on a well-known cellular structure time scheduling technique that has been proposed in [5], [14], [17].

This scheduling technique colors the squares of the cellular structure with finite number of colors, such that every two squares with the same color are farther than a certain distance from each other. Then, it divides the time slide into equal sub-slides where every sub-slide corresponds to one color. Then, it allows the nodes inside squares with the same color transmit simultaneously. The distance between the cells with the same color is large enough so that the interference does not cause any unsuccessful transmission.

The technique guarantees that if transmission power is large enough and $\alpha > d$, then it can provide input/output rate of $\Theta(W)$ for each square of the given cellular structure.

Step 3 (Proof of achievability): We assume that the multicast bits are generated at rate $\lambda'_{\rm m}$ in the network. Next, we compute the average workload that our multicast scheme creates on each cell or large-cell in terms of $\lambda'_{\rm m}$. This gives us the maximum rate of multicast bits that can be delivered in the network successfully under our scheme. Note that the network traffic pattern is symmetrical, therefore the workload is equal (up to a constant factor) over all parts of the network.

First, we analyze the traffic load on the highways. Every multicast packet is repeated in short-hops along highways $\Theta\left(\sqrt{k}\cdot\frac{\sqrt{n}}{c\log(n)}\cdot\log(n) + k\cdot\frac{\sqrt{n}}{c\log(n)\sqrt{k}}\cdot\log(n)\right) = \Theta(\sqrt{nk})$ times a.s.. So, the average workload on each cell on the highway is $\Theta\left(\lambda'_{\rm m}\sqrt{nk}/(\gamma n)\right) = \Theta(\lambda'_{\rm m}\sqrt{\frac{n}{k}})$. Since, the time scheduling provides $\Theta(W)$ for each cell, $\lambda'_{\mathrm{m}} = O(W\sqrt{\frac{n}{k}})$ is the maximum rate of generation of multicast bits that can be loaded the highways.

Second, we analyze the draining traffic on large-cells. From Lemma 4, the number of terminals in each large-cell in at most $c^2 e \log(n)$. Therefore, the maximum workload on a large-cell is $\Theta\left(\lambda'_{\mathrm{m}} \cdot \frac{k}{n} \cdot c^2 e \log(n) \cdot \sqrt{\log(n)}\right) = \Theta\left(\lambda'_{\mathrm{m}} \frac{k\sqrt{\log(n)^3}}{n}\right)$. Since time scheduling provide $\Theta(W)$ for each large-cells, $\lambda'_{\mathrm{m}} = O(W\frac{n}{k\sqrt{\log(n)^3}})$ is the maximum rate of generation of multicast bits that can be drained to the terminals.

If $k = O(\frac{n}{\log(n)^3})$, the highway traffic load is the bottleneck multicast scheme. So $\lambda'_{\mathrm{m}} = \Theta(W\sqrt{\frac{n}{k}})$ is the maximum achievable rate.

Case 2: We apply the same scheme explained in Case 1. However, in this case the draining traffic becomes the bottleneck. Therefore, $\lambda'_{\mathrm{m}} = \Theta(W\frac{n}{k\sqrt{\log(n)^3}})$ is the maximum achievable rate for this case.

Case 3: When $k = \Omega(\frac{n}{\log(n)^2})$, we get better throughput by not using the highway technique of Case 1. Instead, here, we fix the transmission range to r_l and route the packet through large-cells using routing protocol of [22]. It can provide throughput of $\Theta(W\sqrt{\frac{n}{k\log(n)}})$.

Case 4: We can achieve the bound by applying broadcast schemes explained in [13], [15]. Interestingly for achieving this bound the transmission range can be any $r = \Omega(r_l)$. ■

Note that the proposed scheme in Case 1 and 2 can work for all channel models except Protocol Model 3, because, they can vary the transmission range of the nodes at different time slots, however, Protocol Model 3 considers a fixed transmission range.

Next, we combine the computed lower bounds with the upper bounds in (17), then we obtain the following results:

(i) For $d = 1$ dimensional space $\lambda_{\mathrm{m}} = \Theta(W)$

(ii) For $d \geq 2$ dimensional space

$$\lambda_{\mathrm{m}} = \begin{cases} \Theta\left(\sqrt[d]{(\frac{n}{k})^{d-1}}W\right) & \text{if } k \leq \frac{n}{\log(n)^{d+1}} \\ \Omega(\frac{n}{k\sqrt[d]{\log(n)^{d+1}}}W) & \text{if } \frac{n}{\log(n)^{d+1}} \leq k \leq \frac{n}{\log(n)^2} \\ \Omega\left(\sqrt[d]{(\frac{n}{k\log(n)})^{d-1}}W\right) & \text{if } \frac{n}{\log(n)^2} \leq k \leq \frac{n}{\log(n)} \\ \Theta(W) & \text{if } k \geq \frac{n}{\log(n)} \end{cases} \quad (19)$$

The maximum difference between computed upper bound in (17) and lower bound (19) is $O(\sqrt[d]{\log(n)})$ which occurs for $\frac{n}{\log(n)^d} \leq k \leq \frac{n}{\log(n)^2}$.

VI. Multicast Capacity of Large Mobile Networks

In this section we study the multicast capacity of large mobile wireless networks. We consider a stationary mobility model for the nodes with uniform spacial distribution, i.e. we assume that n nodes are moving randomly and mutually independent in a d-dimensional cube with volume V such that at any given time instant the distribution of nodes is uniform.

The capacity mobile wireless networks for unicast flows has been computed in [8]. The paper proves that the aggregate unicast capacity of the nodes grows proportional to n as $\Theta(nW)$. While, [13] shows that broadcast capacity of mobile networks is $\Theta(W)$ and does not change asymptotically with n. Here, we study the multicast capacity of large mobile networks which certainly lies between these two extreme cases asymptotically.

In Theorem 4, we prove that $\lambda_{\mathrm{m}} = \Theta(\frac{n}{k}W)$ for large mobile wireless networks. This is in agreement with the previous results on unicast capacity and broadcast capacity of mobile networks [8], [13]. Note that the formula also demonstrate that the multicast capacity of mobile networks is larger than the capacity static homogeneous networks by at least factor of $\Omega(\sqrt[d]{\frac{n}{k}})$.

Moreover, the theorem demonstrate that the gain of mobility on multicast capacity reduces by increasing the number of terminals in a fixed size mobile network. In extreme case $k = n$ which corresponds to broadcast the mobility gain is bounded by a constant factor.

Theorem 4: Assume a mobile homogeneous wireless network in d-dimensional space. Then for all channel models

$$\lambda_{\mathrm{m}} = \Theta(\frac{n}{k}W) \quad (20)$$

almost surely for large n.

Proof of Theorem 4: We consider an arbitrary node in the network. The rate of infomation which is received or sent by the node is bounded by the wireless channel capacity W ($O(W)$ for Generalized Physical Model). Now, if the multicast bits are generated at rate of λ_{m}, then on average $\lambda_{\mathrm{m}} \cdot \frac{k}{n}$ bits per second must be received/sent by the nodes. It follows that

$$\lambda_{\mathrm{m}} \cdot \frac{k}{n} \leq W \quad (21)$$

Thus, $\lambda_{\mathrm{m}} = O(\frac{n}{k}W)$.

Next, we prove that $\lambda_{\mathrm{m}} = \Theta(\frac{n}{k}W)$ is achievable using a mobility-based routing scheme. We employ a routing scheme similar to the scheme of [8]. A transmission range of $r = \sqrt[d]{\frac{V}{n}}$ is considered for all nodes. A source node transmits the multicast packet to a terminal (which does not have the packet) when they randomly move into the transmission range of each other. For speeding the packet dissemination process, we can allow the terminals which have received the multicast packet act as the source.

Here, a time scheduling technique similar to the proof of Theorem 3 is used for scheduling the transmissions. It guarantees that information bits can be sent to the air and received successfully at the rate of $\Theta(nW)$. Since, the scheme uses at most $k-1$ transmissions for transporting a multicast packet. This scheme can deliver the multicast bits up to the rate of $\Theta(nW/k)$. ■

VII. Conclusion and Future Work

Most existing work on multicast capacity of large homogeneous networks is based on a simple model for wireless channel, namely the Protocol Model [12], [19], [22]. In this paper, we exploit a local capacity tool called *arena* which we introduced recently [14] in order to render multicast accessible to analysis also under more realistic, and notably less pessimistic channel models. Through this study we find three regimes of the multicast capacity (λ_{m}) for large homogeneous wireless network depending on the ratio of terminals over the overall size of the network. Further, the upper bounds of λ_{m} we find are of the order $\sqrt{\log(n)}$ larger than the existing bounds, and we are able to propose a multicast routing and time scheduling scheme to achieve the computed asymptotic bound over all channel models except the simple Protocol Model. To this end, we employ percolation theory among other analytical tools.

Furthermore, we studied the multicast capacity of large mobile wireless networks. We showed that similar to unicast case mobility increases the capacity of wireless networks for multicast asymptotically. However, the mobility gain decreases when increasing the ratio of the number terminals to overall size of the network. In the extreme case where multicast is equivalent to broadcast, the mobility gain reduces to a constant factor.

Future studies can extend the analytical results on multicast capacity for static wireless networks with multi-channels [18] or directional antenna [26]. Also, they can investigate capacity and delay tradeoff for the multicast capacity of mobile wireless network similar to previous studies on unicast capacity [23].

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APPENDIX

A. Completion of the Proof of Theorem 2

We consider one of $3^d - 1$ points and denote it by X. Assume that the set of transmissions $\{(S_{i_1}, D_{i_1}), \ldots, (S_{i_m}, D_{i_m})\}$ contain the point X in their transmission arena. In other words, $|S_{i_j} - X| \leq |S_{i_j} - D_{i_j}|$ for $j = 1, 2, \ldots, m$.

Without lack generality, assume that S_{i_1} is the closet transmitter to X (i.e. $|S_{i_1} - X| \leq |S_{i_j} - X|$).

Next, we bound the sum of the rates of these transmissions and prove that $M = O(1)$ for this particular case.

$$
\begin{aligned}
\sum_{j=1}^{n} W_{i_j} &= \sum_{j=1}^{n} W \log_2(1 + \mathrm{SINR}_{i_j}) \\
&< W_{i_1} + \sum_{j=2}^{n} W \log_2(e) \cdot \mathrm{SINR}_{i_j} \\
&< W_{i_1} + W \log_2(e) \sum_{j=2}^{n} \frac{P_{i_j} |S_{i_j} - D_{i_j}|^{-\alpha}}{\sum_{l \neq j} P_{i_l} |S_{i_l} - D_{i_j}|^{-\alpha}} \\
&\leq W_{i_1} + W \log_2(e) \sum_{j=2}^{n} \frac{P_{i_j} |S_{i_j} - X|^{-\alpha}}{\sum_{l \neq j} P_{i_l} (\sqrt{d+15} |S_{i_l} - X|)^{-\alpha}} \\
&\leq W_{i_1} + W \log_2(e) \sqrt{d+15}^{\alpha} \frac{P_{\max}}{P_{\min}} \sum_{j=2}^{n} \frac{|S_{i_j} - X|^{-\alpha}}{\sum_{l \neq j} |S_{i_l} - X|^{-\alpha}} \\
&< W_{i_1} + W \log_2(e) \sqrt{d+15}^{\alpha} \frac{P_{\max}}{P_{\min}} \sum_{j=2}^{n} \frac{2 |S_{i_j} - X|^{-\alpha}}{\sum_{l=1}^{n} |S_{i_l} - X|^{-\alpha}} \\
&= W_{i_1} + W \log_2(e) \sqrt{d+15}^{\alpha} \frac{P_{\max}}{P_{\min}} \frac{2 \sum_{j=2}^{n} |S_{i_j} - X|^{-\alpha}}{\sum_{l=1}^{n} |S_{i_l} - X|^{-\alpha}} \\
&< W_{i_1} + 2W \log_2(e) \sqrt{d+15}^{\alpha} \frac{P_{\max}}{P_{\min}} \\
&= O(W)
\end{aligned}
$$