Diss. ETH No. 10077

# An Improved Multifractal Formalism and Self-Affine Measures

A dissertation submitted to the
SWISS FEDERAL INSTITUTE OF TECHNOLOGY
ZURICH

for the degree of
Doctor of Mathematics

presented by
RUDOLF HERMANN RIEDI
Dipl. Math. ETH
born June 18, 1961 in CH – St. Gallen
citizen of Switzerland

accepted on the recommendation of
Prof. Dr. Christian Blatter, Examiner
Prof. Dr. Corneliu Constantinescu, Co-Examiner

1993

# Contents

# Introduction

It has been recognized that most fractals $K$ observed in nature are actually composed of an infinite set of interwoven subfractals. This structure becomes apparent when a particular probability measure $\mu$ supported by $K$ is considered: to every $\alpha$ belongs the set $C_\alpha$ of all points of $K$, for which the measure of the balls with radius $\rho$ roughly scales as $\rho^\alpha$ for $\rho \to 0$. In other words the various $C_\alpha$ are the sets of *local Hölder exponent* $\alpha$. Since they are often fractals, $\mu$ was termed *multifractal*.

The complexity of the geometry of $C_\alpha$ is measured by the *spectrum* $f(\alpha)$ which can be thought of as representing the box dimension of $C_\alpha$. More precisely speaking the number of boxes $B$ of a $\delta$-grid with $\mu(B) \simeq \delta^\alpha$ scales as $\delta^{-f(\alpha)}$. However, the singularities of $\mu$ may also be measured through the *generalized dimensions* $d_q$, which are related to the scaling law of the partition sums: $\sum \mu(B)^q \sim \delta^{(q-1)d_q}$.

Spectrum and generalized dimensions are very helpful when comparing multifractals appearing in nature with analytically treatable measures. In various fields multifractality has been found to be appropriate to describe phenomena in nature, like catalytic reactions [GS], the distribution of galaxies [Sa], percolation, Brownian motion [Fed], growing structures [V2, TV], and many others appearing in the theory of dynamical systems [Tél, V1].

Heuristical arguments suggest a close relation between $d_q$ and $f$: the convex $\tau(q) = (1-q)d_q$ is the Legendre transform of the concave $f(\alpha)$. This allows to reduce the somewhat tedious, if not impossible computation of $f$ to the simpler one of $d_q$.

Though used in different fields, the various notions developed for the purpose of numerical simulations differ only slightly. A mathematically precise definition as well as the important relation $\tau(q) = \sup(f(\alpha) - q\alpha)$ can be found in [Falc4]. But unfortunately this concept turns out to be *unsatisfactory* for two reasons.

First of all the spectrum is defined through a double limes, which usually does not exist for great $\alpha$. Secondly the generalized dimensions take the irrelevant value $d_q = \infty$ for negative $q$. More concretely, for as simple multifractals as the middle third Cantor measure half of the singularity exponents are lost and, even worse, the important Legendre relation cannot be verified by [Falc4, prop. 17.2].

In the present thesis we propose a concept which meets the two mentioned problems by a simple improvement. Instead of boxes taken from a grid of size $\delta$, we use a kind of parallel body of these boxes. This renders a measurement $F$ resp. $D_q$ of the singularities of $\mu$, which carries relevant information on the measure and which can be treated with rigorous geometrical arguments. Furthermore, as an invaluable tool for analytical and practical treatment of the spectrum, we introduce the semispectra $F^+$ and $F^-$. Doing so avoids problems of convergence. Moreover, the semispectra are in some sense more regular than the spectrum itself.

The superiority of our concept is reflected in the following facts.

First, the generalized dimensions $D_q$ and the semispectra $F^\pm$ are invariant under bilipschitzian coordinate transformations. Moreover, they depend regularly on the size $\delta$ of the considered grids. In particular it is enough to consider the limit behaviour for any sequence $\delta_n$ such that $\delta_n \geq \delta_{n+1} \geq \nu\delta_n$ with constant $\nu$.

Secondly, the modification does not affect the generalized dimensions for positive $q$, i.e. $D_q = d_q$ $(q \geq 0)$.

Thirdly, the so-called singularity exponents $T(q) := (1-q)D_q$ are indeed the Legendre transform of the spectrum $F(\alpha)$. Therefore $T$ is convex. And, what is even more interesting in applications, if $T$ is continuously differentiable, then $F$ is the Legendre transform of $T$. Similar properties have been claimed for $f$ resp. $d_q$, but hold only for positive $q$.

Multiplicative cascades generalize the construction of the middle third Cantor set. Starting with a compact set $V$, one chooses first $r$ closed disjoint subsets $V_i$ of $V$, then $r$ closed disjoint subsets $V_{i_1 i_2}$ of each $V_{i_1}$ and so on. Provided the diameters of the sets $V_{i_1 \ldots i_n}$ tend to zero with increasing $n$, this process generates a sequence of compact sets $K_n$, each consisting of $r^n$ components, which decreases to a nonempty compact set $K$. Consequently $K$ is homeomorphic to the product space $\{1, \ldots, r\}^{\mathbb{N}}$. Now, given $r$ positive numbers $p_i$ with $p_1 + \ldots + p_r = 1$, there is a unique product measure corresponding to the measures $\{i\} \mapsto p_i$ on the factors of $\{1, \ldots, r\}^{\mathbb{N}}$. Its pullback $\mu$ is a probability measure supported by $K$. The construction of $\mu$ explains the term 'multiplicative cascade'.

It is almost evident that the structure of the cascade must be reflected by the spectrum. Moreover, one's intuition should be that the so-called cylindrical sets $V_{i_1 \ldots i_n}$, which possess the measure $p_{i_1} \cdot \ldots \cdot p_{i_n}$, give the essential information about the geometry of $\mu$—at least if their shapes are similar.

As a consequence most of the notions of spectrum used for the study of multiplicative cascades work directly with coverings of $K$ by cylindrical sets, instead of coverings by boxes from a grid. Such a formalism may rightly be called 'tailored to multiplicative cascades'. It certainly possesses great advantages, since the embedding of $\mu$ in the particular euclidean space causes no problem. Furthermore symbolic dynamics are used most effectively [BR, CM]. For instance the spectrum defined through cylindrical sets is always concave [Lan]. Furthermore, for self-similar measures, i.e. $\mu = \sum p_i w_{i*}\mu$ with similarities $w_i$, the spectrum can be calculated with reasonable effort [HP]. On the other hand this multifractal formalism cannot distinguish between $\mu$ and the product measure. In other words it lacks geometrical relevance and cannot be used to *detect* structures.

But exactly this is the aim of our approach. In chapter 1 we develop a multifractal



formalism which is based purely on the geometry of measures and which does not assume any stucture in advance.

Consequently we will treat arbitrary probability measures with bounded support. Moreover, we will not assume concavity nor differentiability of the spectrum $F$. To support this approach examples will be developed, for which these properties do not hold. So the duality between $T(q)$ and $F(\alpha)$ is violated: the singularity exponents depend more regularly on the measures of the boxes, since they are defined through a sum or 'average' [HJKPS]. The spectrum, on the other hand, carries a greater amount of information.

The present formalism can be applied to any probability measure, in particular to one obtained from observation and represented—incompletely of course—by some thousands of sample points. Since the multifractal formalism can be used to compare such an observed measure with an analytically treatable multifractal, we provide formulas for the spectrum of self-similar measures in chapter 2 and of certain self-affine measures in chapter 3.

A satisfying result is that the improved formalism leads to the same conclusions as the one 'tailored to multiplicative cascades'—at least for self-similar measures. In this case all the cylindrical sets have the same shape and can be approximated by balls. However, in the affine case, some of the cylindrical sets are long stretched and thin. These sets of course cannot be thought of as representing balls. Our treatment of this case shows how to modify one's intuition.

Finally, besides a lot of examples, we provide evidence as well as rigorous results concerning the interpretation of $F(\alpha)$ as the Hausdorff dimension of the sets $C_\alpha$ of local Hölder exponent $\alpha$. The main problem here is, that the boxes of size $\delta$ with measure $\sim \delta^\alpha$ do not necessarily form a sequence of decreasing sets as $\delta \to 0$.

Summarizing we feel that the present thesis provides a better understanding of multifractality and refines one's intuition on self-affine measures.

# Abstract


To characterize the geometry of a probability measure $\mu$ with bounded support, its so-called spectrum $f$ has been introduced recently. A mathematically precise definition has been given in [Falc4]:

$$f(\alpha) = \lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\frac{\log(n_\delta(\alpha+\varepsilon) - n_\delta(\alpha-\varepsilon))}{-\log\delta}$$

whenever this limes exists. Thereby $n_\delta(\alpha)$ is the number of boxes $B = \prod[l_k\delta,(l_k+1)\delta[$ in $\mathbb{R}^d$ with integers $l_k$, such that $\mu(B) \geq \delta^\alpha$. As will be shown, this definition is unsatisfactory for reasons of convergence as well as of undesired sensitivity to the particular choice of coordinates. A new definition $F$ of the spectrum is introduced, which is based on box-counting too, but which carries relevant information about $\mu$. The essential modification is that $n_\delta$ is replaced by the number of boxes $N_\delta(\alpha)$ with $\mu((B)_1) \geq \delta^\alpha$, where $(B)_1$ is the box of size $3\delta$ concentric to $B$. In addition, the $\lim_{\delta\to 0}$ is replaced by the $\limsup_{\delta\to 0}$ for obvious reasons. The adaptation of the well known singularity exponents to this concept reads:

$$T(q) = \limsup_{\delta\downarrow 0}\frac{\log\left(\sum\mu((B)_1)^q\right)}{-\log\delta}.$$

This notion renders exponents $T(q)$, which are invariant under bi-lipschitzian coordinate transformations and for which the limit behaviour can be extracted from considering any sequence $\delta_n$ such that $\delta_n \geq \delta_{n+1} \geq \nu\delta_n$ with constant $\nu$.
The important relation

$$T(q) = \sup_{\alpha\in\mathbb{R}}\left(F(\alpha) - q\alpha\right)$$

is valid for $q \neq 0$ and in the case of multiplicative cascades also for $q = 0$. Consequently $T(q)$ is convex. On the other hand, $F(\alpha)$ need not be concave, as examples prove. In other words $F$ may provide more detailed information than the Legendre transform of $T$. However, if $T(q)$ equals $\lim_{\delta\to 0}\ldots$ and is differentiable on all of $\mathbb{R}$, then

$$F(\alpha) = \lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\frac{\log(N_\delta(\alpha+\varepsilon) - N_\delta(\alpha-\varepsilon))}{-\log\delta} = \inf_{q\in\mathbb{R}}(T(q) + q\alpha)$$

for all $\alpha$.

Invariant measures play a crucial role in multifractal theory. They satisfy an invariance condition $\mu = \sum_1^r p_i w_{i*}\mu$ with positive numbers $p_i$ such that $p_1 + \ldots + p_r = 1$. When the maps $w_i$ are similitudes, i.e. $|w_i(x) - w_i(y)| = \lambda_i|x-y|$, and when the open set condition (OSC) holds, then $\mu$ is termed self-similar and the singularity exponents satisfy

$$\sum_{i=1}^{r} {p_i}^q {\lambda_i}^{T(q)} = 1 \tag{1}$$

for all $q \in \mathbb{R}$. This formula is already well known, but only for positive $q$. Moreover, no rigorous proof has been given until now.
Self-affine measures are defined mutatis mutandis. We investigate maps of the form $w_i(x,y) = (\pm\lambda_i x + u_i, \pm\nu_i y + v_i)$. Again under certain open set conditions we prove that $T(q) = \max(\Gamma^+(q), \Gamma^-(q))$. Thereby $\Gamma^+$ and $\Gamma^-$ can be obtained from equations which naturally involve the characteristical values $\lambda_i$ and $\nu_i$ and which reduce to (1) if $\lambda_i = \nu_i$. In particular, the box dimension of the support $K$ is recovered for $q = 0$:

$$d_{\text{box}}(K) = T(0) = \max(d^+, d^-).$$

Thereby $d^+$ and $d^-$ are defined through

$$\sum_{i=1}^{r} {\lambda_i}^{D^{(1)}} {\nu_i}^{(d^+ - D^{(1)})} = 1 \quad \text{resp.} \quad \sum_{i=1}^{r} {\nu_i}^{D^{(2)}} {\lambda_i}^{(d^- - D^{(2)})} = 1,$$

and $D^{(k)}$ denotes the box dimension of the projection of $K$ onto the $x^{(k)}$-axis. Finally, the 'almost sure' Hausdorff dimension of $K$ [Falc3] equals

$$d_{HD}(K) = \max(\Delta^+, \Delta^-),$$

where

$$\left\{ \begin{array}{ll} \sum_{i=1}^{r} \lambda_i^{\Delta^+} = 1 & \text{if } \sum \lambda_i \le 1, \\ \sum_{i=1}^{r} \lambda_i \nu_i^{\Delta^+ - 1} = 1 & \text{otherwise,} \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{ll} \sum_{i=1}^{r} \nu_i^{\Delta^-} = 1 & \text{if } \sum \nu_i \le 1, \\ \sum_{i=1}^{r} \nu_i \lambda_i^{\Delta^- - 1} = 1 & \text{otherwise.} \end{array} \right\}$$

# Zusammenfassung

Um die Geometrie eines Wahrscheinlichkeitsmasses $\mu$ mit beschränktem Träger charakterisieren zu können wurde der Begriff des 'Spektrums' $f$ eingeführt. Eine mathematisch präzise Definition findet sich in [Falc4]:

$$f(\alpha) = \lim_{\varepsilon\downarrow 0} \lim_{\delta\downarrow 0} \frac{\log(n_\delta(\alpha+\varepsilon) - n_\delta(\alpha-\varepsilon))}{-\log\delta}$$

wann immer dieser Grenzwert existiert. Dabei ist $n_\delta(\alpha)$ die Anzahl der Würfel $B = \prod[l_k\delta, (l_k+1)\delta[$ in $\mathbb{R}^d$ mit ganzzahligen $l_k$, für welche $\mu(B) \ge \delta^\alpha$. Wie gezeigt wird, ist diese Definition unbefriedigend aus Gründen der Konvergenz wie auch wegen einer unerwünschten Abhängigkeit von der Wahl der Koordinaten. Ein neues Spektrum $F$ wird eingeführt, welches ebenfalls auf 'box counting' beruht, aber geometrisch relevante Informationen trägt. Die wesentliche Veränderung besteht darin, dass neu statt $n_\delta$ die Anzahl $N_\delta(\alpha)$ der Würfel mit $\mu((B)_1) \ge \delta^\alpha$ verwendet wird, wobei $(B)_1$ der zu $B$ konzentrische Würfel mit Seite $3\delta$ ist. Aus einleuchtenden Gründen wird zusätzlich der $\lim_{\delta\to 0}$ ersetzt durch $\limsup_{\delta\to 0}$. Die wohlbekannten 'Singularitäts Exponenten' werden in dieses Konzept eingebettet durch die Definition

$$T(q) = \limsup_{\delta\downarrow 0} \frac{\log\left(\sum \mu((B)_1)^q\right)}{-\log\delta}.$$

Sie sind invariant unter bilipschitz-stetigen Koordinatentransformationen, und das massgebende asymptotische Verhalten kann aus einer beliebigen Folge $\delta_n$ herausgelesen werden, vorausgesetzt $\delta_n \ge \delta_{n+1} \ge \nu\delta_n$ mit konstantem $\nu$.
Die wichtige Beziehung

$$T(q) = \sup_{\alpha\in\mathbb{R}} \left(F(\alpha) - q\alpha\right)$$

ist allgemein gültig für $q \ne 0$, und typischerweise auch für $q = 0$. Folglich ist $T(q)$ konvex. Andererseits muss $F(\alpha)$ keineswegs konkav sein, wie Beispiele belegen. In anderen Worten, $F(\alpha)$ kann mehr Information über $\mu$ beinhalten als die Legendre Transformierte von $T(q)$. Doch solche Beispiele scheinen untypisch zu sein, denn es gilt: Falls $T(q)$ gleich $\lim_{\delta\to 0}\ldots$ und differenzierbar ist überall auf $\mathbb{R}$, dann ist

$$F(\alpha) = \lim_{\varepsilon\downarrow 0} \lim_{\delta\downarrow 0} \frac{\log(N_\delta(\alpha+\varepsilon) - N_\delta(\alpha-\varepsilon))}{-\log\delta} = \inf_{q\in\mathbb{R}} (T(q) + q\alpha)$$

für alle $\alpha$.
Invariante Masse spielen eine zentrale Rolle in der Theorie der Multifraktale. Sie erfüllen eine Invarianzbedingung der Art $\mu = \sum_1^r p_i w_{i*}\mu$, wobei die $p_i$ positive Zahlen sind mit $p_1 + \ldots + p_r = 1$. Falls die Abbildungen $w_i$ Ähnlichkeiten sind, also falls $|w_i(x) - w_i(y)| = \lambda_i|x - y|$, and falls es eine 'Urzelle' gibt — englisch: open set condition (OSC) — dann nennt man $\mu$ selbst-ähnlich, und die Singularitäts Exponenten erfüllen die Gleichung

$$\sum_{i=1}^{r} {p_i}^q {\lambda_i}^{T(q)} = 1 \tag{1}$$

für alle $q \in \mathbb{R}$. Diese Formel ist bereits bekannt, jedoch nur für positive $q$. Ausserdem ist uns bis jetzt kein strenger Beweis bekannt.
Selbst-affine Masse werden mutatis mutandis definiert. Wir untersuchen Abbildungen der Form $w_i(x, y) = (\pm\lambda_i x + u_i, \pm\nu_i y + v_i)$. Wiederum unter gewissen 'Urzellenbedingungen' gilt $T(q) = \max(\Gamma^+(q), \Gamma^-(q))$. Dabei können $\Gamma^+$ und $\Gamma^-$ aus Gleichungen erhalten werden, welche die charakteristischen Werte $\lambda_i$ und $\nu_i$ natürlich einbeziehen, und welche sich im Falle $\lambda_i = \nu_i$ auf (1) vereinfachen. Insbesondere erhält man die Box Dimension des Trägers $K$ für $q = 0$:

$$d_{\text{box}}(K) = T(0) = \max(d^+, d^-).$$

Dabei sind $d^+$ und $d^-$ definiert durch

$$\sum_{i=1}^{r} {\lambda_i}^{D^{(1)}} {\nu_i}^{(d^+ - D^{(1)})} = 1 \quad \text{und} \quad \sum_{i=1}^{r} {\nu_i}^{D^{(2)}} {\lambda_i}^{(d^- - D^{(2)})} = 1,$$

und $D^{(k)}$ steht für die Box Dimension der Projektion von $K$ auf die $x^{(k)}$-Axe. Schliesslich lässt sich die 'fast sichere' Hausdorff Dimension von $K$ [Falc3] berechnen durch

$$d_{HD}(K) = \max(\Delta^+, \Delta^-),$$

wobei

$$\left\{ \begin{array}{ll} \sum_{i=1}^{r} \lambda_i^{\Delta^+} = 1 & \text{falls } \sum \lambda_i \le 1, \\ \sum_{i=1}^{r} \lambda_i \nu_i^{\Delta^+ - 1} = 1 & \text{sonst,} \end{array} \right\} \quad \text{und} \quad \left\{ \begin{array}{ll} \sum_{i=1}^{r} \nu_i^{\Delta^-} = 1 & \text{falls } \sum \nu_i \le 1, \\ \sum_{i=1}^{r} \nu_i \lambda_i^{\Delta^- - 1} = 1 & \text{sonst.} \end{array} \right\}$$

# Chapter 1

# A Multifractal Formalism

To characterize the geometry of a measure, its so-called spectrum has proved to be an invaluable tool. The present chapter, dedicated to its study, is organized as follows: in the first section we introduce the multifractal formalism based on box-counting—as it is developed and used at present—and give two reasons why it should be changed. Thereby we stick to the notation of Falconer [Falc4]. In section two we present a new concept and show that it possesses the expected regularities. In the third section simple properties of the newly defined spectrum are discussed, in particular its connection to the so-called singularity exponents.

The spirit of our approach is summarized in two remarks. First, we shall consider arbitrary probability measures with compact support and provide results of considerable generality. However, when more is assumed about the measure, in particular that it is constructed by a multiplicative cascade, much more can be said. This will be carried out in the remaining chapters. Secondly, we focus in this work on a formalism based purely on the geometry of the measure, allowing a characterization free from any assumption on the structure in advance. This is quite different from a concept which emphasizes on multiplicative cascades and uses this underlying structure essentially. However for the measures considered in the subsequent chapters, the two formalisms lead to identical conclusions. This will allow us to study limit behaviours in the more convenient space $\{1, \ldots, r\}^{\mathbb{N}}$ and to derive exact formulas for the spectra.

## 1.1 The Status Quo

A compact set $K$ in euclidean space $\mathbb{R}^d$, such as the middle third Cantor set, may carry a rich geometrical structure. One way to measure the complexity of its geometry is to use a *δ-grid* with variable $\delta > 0$ in the following way: A set of the form

$$\prod_{k=1}^{d}[l_k\delta, (l_k+1)\delta[$$

with integer values $l_k$ for $k = 1, \ldots, d$ will be called a *$\delta$-box*. Letting $\mathcal{N}_\delta(K)$ denote the number of $\delta$-boxes meeting $K$, the *box dimension* of $K$ is defined as

$$\overline{d_{\text{box}}}(K) = \limsup_{\delta\downarrow 0} \frac{\log \mathcal{N}_\delta(K)}{-\log\delta}.$$

Moreover, the notation $d_{\text{box}}(K)$ is used to indicate that the $\lim_{\delta\to 0}$ actually exists. The box dimension measures the amount of information needed to locate a point on $K$. In particular, the box dimensions of a point, a line and a square are 0, 1 and 2, respectively, which explains the name dimension. The literature on this field is vast. For a profound introduction and for different equivalent definitions we recommend [Falc4].

The dimension $d_{\text{box}}(K)$ describes the geometry of $K$ in a global manner. More subtle structures of $K$ can be detected by considering an appropriate probability measure $\mu$ with support $K$. As an easy example imagine $K$ to be the union of a line segment and a disjoint square and let the measure $\mu_{ls}$ correspond to length on the line and to area on the square. Then, obviously, $\mu_{ls}$ is more strongly 'concentrated' on the line than on the square. In general one may think of $K$ as the union of infinitely many interwoven subsets, usually fractals, with homogeneous concentration of $\mu$. Since this structure is induced by $\mu$ we call $\mu$ a *multifractal*. Thus, we use 'multifractal' as a synonym for probability measure with compact support. Note, however, that other authors [MEH, HJKPS] use the same name with different meanings.

A first attempt to seize this structure is the following, given by Falconer [Falc4, p 257]:

**Definition 1.1 (Falconer)** *Let $G_\delta := G_\delta^\mu$ be the set of all $\delta$-boxes with $\mu(B) \neq 0$ and let $n_\delta(\alpha)$ be the number of all boxes in $G_\delta$ with $\mu(B) \geq \delta^\alpha$. Then*

$$f(\alpha) := \lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} \frac{\log(n_\delta(\alpha+\varepsilon) - n_\delta(\alpha-\varepsilon))}{-\log\delta}$$

*whenever the limes exists. Thereby $\log(0) := -\infty$ and this value is allowed for $f$.*

This definition is still based on the method of box counting, but it involves the measure $\mu$. The function $f$ is usually referred to as the *multifractal spectrum* or simply *spectrum* of $\mu$. For the simple example $\mu_{ls}$ above $f$ takes only two nontrivial values, namely $f(1) = 1$ and $f(2) = 2$, which arise from the boxes covering the straight line and the square, respectively. So, given $\alpha$, the value of $f(\alpha)$ indicates the density in the sense of box-counting of the set of points where $\mu$ has the 'concentration' $\alpha$. This is the intuitive understanding of the spectrum [HJKPS, JKL]. Indeed, under certain conditions $f(\alpha)$ equals the dimension of the set of all points



$x$ with *local Hölder exponent* $\alpha$, which means that the measure of a ball with center $x$ scales as the $\alpha$-th power of its diameter [EM]. For more precise statements we refer to example 2.11 and to [CM, CLP, S]. Furthermore, the spectrum $f$ of certain self-similar measures can be directly related to the (probabilistic) distribution of the local Hölder exponents [EM]. But note that this matter is far from trivial, if treated rigorously.

The spectrum often is a strictly concave function with a single maximum. But it may as well look quite different (see Ex. 2.15, 2.16, 2.17, 3.3, 3.5 and 3.6). Note that most proofs in the literature [BR, Lan] concerning this matter work with a notion of spectrum 'tailored for multiplicative cascades' rather than one relying on box-counting.

However, provided $f$ is concave, then $f(\alpha) = \sup\{f(t) : t < \alpha\}$ in the increasing part and $f(\alpha) = \sup\{f(t) : t > \alpha\}$ in the decreasing part of $f$. This leads us to defining the following auxiliary functions:

**Definition 1.2** *Let $m_\delta(\alpha)$ be the number of all boxes in $G_\delta$ with $\mu(B) < \delta^\alpha$, so that $n_\delta(\alpha) + m_\delta(\alpha)$ equals $\#G_\delta$, the number of all $\delta$-boxes with nonvanishing measure. Then set*

$$f^+(\alpha) = \limsup_{\delta\downarrow 0} \frac{\log n_\delta(\alpha)}{-\log\delta} \qquad f^-(\alpha) = \limsup_{\delta\downarrow 0} \frac{\log m_\delta(\alpha)}{-\log\delta}.$$

Despite their simplicity, $f^+$ and $f^-$ usually contain the same information on $\mu$ as does $f$: when $n_\delta$ is strictly increasing at $\alpha$, the term $n_\delta(\alpha-\varepsilon)$ in the difference $n_\delta(\alpha+\varepsilon) - n_\delta(\alpha-\varepsilon)$ is negligible. So, the increasing part of $f$ is usually equal to $f^+$. Since $n_\delta(\alpha+\varepsilon) - n_\delta(\alpha-\varepsilon) = m_\delta(\alpha-\varepsilon) - m_\delta(\alpha+\varepsilon)$, a similar argument shows that the decreasing part of $f$ is usually equal to $f^-$.

Performing numerical simulations or analytical investigations one will find it hard, if not impossible, to calculate $f^+$ and $f^-$. Even more tedious is the computation of $f$. But the related singularity exponents $\tau(q)$ defined below are easier to determine [GP2, GP1, Gr1, BP, JKP, BPPV, L], in particular since they depend more regularly on the data $\mu(B)$ [HJKPS, JKL]. Because $f(\alpha)$ is usually a concave function related to $\tau(q)$ through the Legendre transformation [CLP, Falc4, BR] it is in most cases enough to know $\tau(q)$.

This points to certain advantages of $f^\pm$ in contrast with $f$: their definition is simple and $f^\pm$ are defined for all $\alpha$. Moreover, it is straightforward that they are monotonous. As a consequence, $f^+$ and $f^-$ can immediately be derived from $\tau(q)$, while the existence and the concavity of $f$ do not hold in general and have to be verified *before* applying the Legendre transformation. (Compare our theorem 1.1 with proposition 17.2 in [Falc4]).

By introducing $f^+$ and $f^-$ a central difficulty in the handling of spectra—**existence and concavity of $f$**—is removed .

To discuss a second difficulty hidden in definition of $f$ let us introduce $\tau(q)$. We keep close to the notation in [Falc4, p 255]:

**Definition 1.3** *For* $q \in \mathbb{R}$ *let* $s_\delta(q) := \sum_{B\in G_\delta} \mu(B)^q$ *and define the* singularity exponents *to be*

$$\tau(q) = \limsup_{\delta\downarrow 0} \frac{\log s_\delta(q)}{-\log\delta}.$$

*For later convenience, denote the corresponding* $\liminf$ *by* $\underline{\tau}(q)$*. Finally the* generalized dimensions *are given by*

$$d_q := \frac{1}{1-q}\tau(q) \quad \textit{for } q \neq 1 \quad \textit{and} \quad d_1 = \limsup_{\delta\downarrow 0} \sum_{B\in G_\delta} \mu(B)^q \frac{\log\mu(B)}{\log\delta}.$$

The definition of $G_\delta$ guarantees that boxes of measure zero do not contribute to $s_\delta$. Usually (see corollary 2.1) $d_0$ is just $\overline{d_{\text{box}}}(K)$, which explains the name 'generalized dimensions'. Since $\tau(1) = 0$ holds trivially, a closer look at $d_1$ is required. This will be carried out in section 1.3. The special value $d_1$, which seems to be the most interesting of all $d_q$ for several reasons [Gr2], was termed *information dimension*. Together with the correlation dimension $d_2$ it was the first of all $d_q$ to be introduced [GP1, GP2, OWY].

Several recent publications are concerned with the generalized dimensions of concrete examples and include more or less rigorously derived formulas. As it was recognized in [CLP, HR], the difficulty in calculating $d_q$ is imperceptibly hidden in the negative $q$-range: Even for the simplest multifractals as the middle third Cantor set (Ex. 1.1), boxes with exceptionally small measure occur for certain scales $\delta$ of the grid. When risen to a negative power the measures of these boxes give an unnatural large contribution to $s_\delta(q)$. They will dominate the asymptotics of $s_\delta(q)$, which is of course not intended.

A different asymptotic behaviour can only be obtained by restricting the measurement to certain scales $\delta$ of the considered grids. But this requires that the structure to be detected is known in advance. This point will be made explicit in the following example. Note that it is neither exotic nor pathological, but on the contrary the multifractal presented the most.

**Example 1.1 (Cantor Measure)** Let $p_1 > 0$, $p_2 > 0$ with $p_1 + p_2 = 1$. Subdivide $[0,1]$ into three equally spaced intervals, assign the measure $p_1$ to the left one, the measure $p_2$ to the right one and throw away the middle one. With the remaining two intervals proceed the same way, creating four intervals of length $1/9$ so that the first one obtains the measure ${p_1}^2$, the second and the third $p_1 \cdot p_2$ and the last one ${p_2}^2$ (see Fig. 1.1). Repeating this procedure ad infinitum leaves one with a Borel probability measure (compare section 2.1) having the well known middle third Cantor set as its support.

The first claim is:

$$q < 0 \qquad \Rightarrow \qquad \tau(q) = \infty.$$

**Proof** To every $n \in \mathbb{N}$ there is a $k_n \in \mathbb{N}$ with ${p_2}^{k_n} \leq (1/2 \cdot 3^{-n})^n$, because $p_2 < 1$. Without loss of generality $k_n \geq n+1$. Then, $\delta_n := (1 - 3^{-k_n})3^{-n}$ lies in



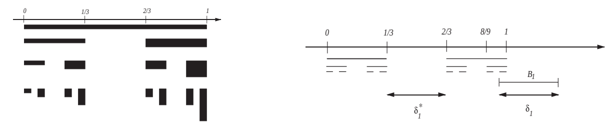


Figure 1.1: *The construction of the Cantor multifractal for* $p_1 = 1/3$ *and* $p_2 = 2/3$ *on the left and an illustration concerning the exceptional behaviour of some* $\delta_n$*-boxes on the right* ($n = 1$)*.*

$[(1 - 3^{-n})3^{-n}, 3^{-n}]$. Since $(3^n + 1)\delta_n \geq 1$, the box $B_n := [3^n\delta_n, (3^n+1)\delta_n[$ has very small measure: $B_n \cap [0,1] = [1 - 3^{-k_n}, 1]$, thus $\mu(B_n) = {p_2}^{k_n} \leq (\delta_n)^n$. For $q < 0$ it follows

$$s_{\delta_n}(q) \geq \mu(B_n)^q \geq (\delta_n)^{nq}$$

and

$$\frac{\log s_{\delta_n}(q)}{-\log\delta_n(q)} \geq n \cdot (-q),$$

which proves the claim. $\diamond$

To keep the proof simple, the box $B_n$ was chosen in special position at the border of $[0,1]$. However, the same behaviour can of course be recognized at every point of $K$.

Only the restriction of $\delta$ to a suitable sequence as e.g. $\delta_n^* := 3^{-n}$ allows to observe the naturally expected behaviour:

$$s_{\delta_n^*}(q) = \sum_{(i_1,\ldots,i_n)\in\{1,2\}^n} p_{i_1}^q \cdot \ldots \cdot p_{i_n}^q = ({p_1}^q + p_2^q)^n,$$

and hence

$$\tau^*(q) := \lim_{n\to\infty} \frac{\log s_{\delta_n^*}(q)}{-\log\delta_n^*} = \frac{\log(p_1^q + p_2^q)}{\log 3}.$$

By the argument of theorem 1.2 the double limes

$$f^*(\alpha) := \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty} \frac{\log(n_{\delta_n^*}(\alpha+\varepsilon) - n_{\delta_n^*}(\alpha-\varepsilon))}{-\log\delta_n^*}$$

exists for all $\alpha \in \mathbb{R}$ and equals the Legendre transform $\inf_{q\in\mathbb{R}}(\tau^*(q) + \alpha q)$ of $\tau^*(q)$. The function $f^*(\alpha)$ carries information about $\mu$ and agrees with the intuitive understanding of the spectrum: it gives the dimension of the sets with Hölder exponent $\alpha$. The graphs of $\tau^*$ and $f^*$ reveal the typical features, in particular the asymptotic behaviour of $d_q^* = \tau^*(q)/(1-q)$ and the concavity of $f^*$ (see Fig. 1.2).

A closer look reveals $\tau^*(q) = \underline{\tau}(q)\ \forall q \in \mathbb{R}$ and even $\tau(q) = \underline{\tau}(q) = \tau^*(q)$ for $q \geq 0$. But note that this result is derived using strongly the particular position of $K$ in the $3^{-n}$-grid.

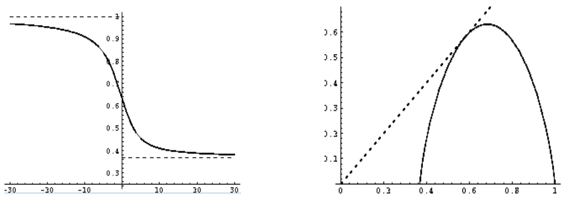

Figure 1.2: *Generalized dimensions (on the left) and spectrum of the middle third Cantor measure with* $p_1 = 2/3$ *und* $p_2 = 1/3$*. The dashed parts are the asymptotical lines for* $d^*$ *resp. the internal bisector of the axes touching the graph of* $f^*$*.*

**Proof** In section 1.2 we will introduce $\underline{T}(q)$ which is a lower bound to $\underline{\tau}(q)$ due to proposition 1.6. By corollary 2.5 $T(q) = \underline{T}(q) = \tau^*(q)$. Moreover, $T(q) = \tau(q)$ for $q \geq 0$ again by proposition 1.6. Regarding the sequence $\delta_n^*$ completes the proof. $\diamond$
Similarly one can show

$$f^*(\alpha) = \lim_{\varepsilon \downarrow 0} \liminf_{\delta \downarrow 0} \frac{\log\left(n_\delta(\alpha+\varepsilon) - n_\delta(\alpha-\varepsilon)\right)}{-\log \delta}$$

for all $\alpha \in \mathbb{R}$, and even $f^*(\alpha) = f(\alpha)$ for $\alpha \leq \alpha_0$, where $\alpha_0$ denotes the maximum of $f^*$. However, provided $\alpha$ is large enough to satisfy $\delta^\alpha < \min(p_1, p_2)$, the difference $n_\delta(\alpha+\varepsilon) - n_\delta(\alpha-\varepsilon)$ takes the value zero as well as positive values for arbitrary small $\delta > 0$. Consequently, $f(\alpha)$ does not exist for these $\alpha$. ○

This example provides strong arguments for the need of a concept replacing definition 1.1:

- For **negative** $q$ the singularity exponents $\tau(q)$ do not provide any useful information about $\mu$.
- As a first alternative $\underline{\tau}(q)$ is at hand. However, even in the simple example 1.1 $\underline{\tau}(q)$ depends heavily on the particular positioning of $\mu$ in the space.
- The double limes $f(\alpha)$ does in general not exist for all $\alpha$. So, the theory developed in [Falc4] cannot be applied (in particular proposition 17.2 giving the Legendre relation between $f$ and $\tau$).
- Finally, one may interfere that the restriction of the considered values $\delta$ to a suitable sequence $\delta_n$ would be sufficient. But the comparison of $\delta_n^*$ and $\delta_n \in [(1-3^{-n})\delta_n^*, \delta_n^*]$ in example 1.1 must destroy any confidence in such practice, since strict self-similarity cannot be expected in numerical simulations. Moreover, even if the latter would be assumed, one had to know *in advance* the contraction ratios of the self-similar process generating the multifractal.



It is our concern to show that a simple but effective change in the way of measuring the concentrations of $\mu$ is enough to make the generalized dimensions a useful tool also for negative $q$.

## 1.2 An Improved Formalism

This section provides the definition as well as the basic properties of the formalism we propose.
Before starting we would like to put our method into a broader context.
Remember that the undesired behaviour described in the previous section arises from boxes with exceptionally small measure. Collet et al. [CLP] pointed out that for certain measures every such box possesses a neighbouring box with 'normal' measure. While in [CLP] a very technical partition sum is constructed to replace $s_\delta(q)$, the idea in our mind is as simple as effective: we use the measure of boxes blown up by a factor three.
The essential geometrical argument in the proofs below will be the following: whenever a box $B$ intersecting $K$ is considered, the enlarged concentric box $B'$ meets $K$ in its 'middle part', i.e./ in $B$. Hence $B'$ is a better approximation of a ball with center in $K$ than the original box $B$ (see Fig. 1.3). So, we feel that this method is

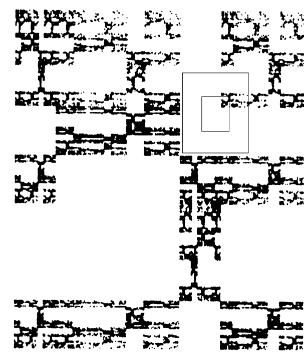

Figure 1.3: *When a box intersects the support* $K$*, then an enlarged and concentric box will constitute a better approximation of a ball centered in* $K$ *and eventually meets* $K$ *in a more representative part than the original box.*

more accurate to measure local behaviour such as the Hölder exponents, where one usually works with balls centered in $K$. Moreover, for multifractals constructed as the Cantor measure, a considerable part of $K$ must be contained in the enlarged box, leading to further properties and to the formulas for the spectrum of self-similar and self-affine measures as given in the subsequent chapters.

However, this does not mean that every multifractal can entirely be described by its spectrum. In particular, also the newly defined singularity exponents may be infinite and so-called left-sided spectra may occur. For examples see example 2.14 and [MEH, ME, GA]. But it is important to notice that with the new concept one can be sure that infinite singularity exponents imply arbitrarily large local Hölder exponents, while in the former formalism $\tau(q) = \infty$ may as well arise from inappropriate measurement.

Our formalism uses the parallel-body of a box: for $\kappa > 0$ and $B = \prod_{k=1}^{d} [l_k\delta, (l_k+1)\delta[$ let

$$(B)_\kappa := \prod_{k=1}^{d} [(l_k - \kappa)\delta, (l_k + 1 + \kappa)\delta[. \tag{1.1}$$

As will be shown, the particular choice of $\kappa$ is of no importance, as long as it is kept fixed through the process. For numerical simulations it might be most convenient to choose $\kappa = 1$.

**Definition 1.4 (Singularity Exponents)** *For $q \in \mathbb{R}$ let $S_\delta(q) := \sum_{B \in G_\delta} \mu((B)_1)^q$ and set*

$$T(q) = \limsup_{\delta \downarrow 0} \frac{\log S_\delta(q)}{-\log \delta}.$$

*Thereby the value $\infty$ is allowed. For convenience denote the respective* lim inf *by $\underline{T}(q)$. Moreover, whenever $T(q)$ and $\underline{T}(q)$ coincide for a particular $q$, $T(q)$ will be called* grid-regular.

Note that the condition '$\mu(B) \neq 0$' chooses the boxes, not '$\mu((B)_1) \neq 0$'. This is the **central idea of the new formalism** (see also Fig. 1.3).

Note, furthermore, that $\underline{T}(q) > -\infty$ for all $q$: if $q < 0$, then $S_\delta(q) \geq 1$. If $q \geq 0$ consider a $\delta$-box $B$ with maximal measure. Of course $\mu(B) \geq 1/\#G_\delta$, and since there are at the most $\#G_\delta \leq c \cdot \delta^d$ boxes with nonvanishing measure, this leads to $\underline{T}(q) \geq -dq$.

The question concerning the grid-regularity of $T$ is of importance in applications for obvious reasons.

The same argument that gives the independence of $T$ from the choice $\kappa = 1$ also proves its invariance under a considerable class of coordinate transformations and justifies the restriction of the considered $\delta$ to a suitable sequence. So, the three assertions will be treated in one proposition. Following the usual lines of interest [OWY, Koh, FM] a bijective map $\Phi$ from an open neighbourhood $U$ of $K = \text{supp}(\mu)$ into $\mathbb{R}^d$ will be called an *admissible* coordinate transformation, if it is bi-lipschitzian, i.e. if $L^{-1}|x-y| \leq |\Phi(x) - \Phi(y)| \leq L|x-y|$ for some constant $L$. A sequence $(\delta_n)_{n\in\mathbb{N}}$ will be called *admissible*, if $\delta_n \to 0$ and if there is a $\upsilon > 0$ such that $\delta_n \geq \delta_{n+1} \geq \upsilon\delta_n \, \forall n \in \mathbb{N}$.



**Proposition 1.5** *Let $\Phi$ be an admissible coordinate transformation, let $\mu' = \Phi_*\mu$, $\kappa' > 0, \kappa > 0$ and let $(\delta_n)_{n\in\mathbb{N}}$ be an admissible sequence. Then*

$$T(q) = \limsup_{n\to\infty} \frac{\log\left(\sum_{B\in G_{\delta_n}} \mu((B)_\kappa)^q\right)}{-\log \delta_n} = \limsup_{\delta\downarrow 0} \frac{\log\left(\sum_{B\in G'_\delta} \mu'((B)_{\kappa'})^q\right)}{-\log\delta}$$

*for all $q \in \mathbb{R}$. Similar for $\underline{T}(q)$. In particular these values are independent of $\kappa$ and of the choice of the coordinate system.*

**Remark** The same holds for $\tau(q)$, but only for $q \geq 0$, as the two admissible sequences of example 1.1 show.

**Proof** We let $G'_{\delta'} = G^{\mu'}_{\delta'}$ be the set of $\delta'$-boxes $B$ with $\mu'(B) \neq 0$ and compare the values $S'_{\delta'}(q,\kappa') = \sum_{B\in G'_{\delta'}} \mu'((B)_{\kappa'})^q$ with $S_\delta(q,\kappa) = \sum_{B\in G_\delta} \mu((B)_\kappa)^q$.

**i)** To every term in $S'_{\delta'}(q,\kappa')$ a greater one will be constructed in $S_\delta(q,\kappa)$.

Take $\delta' > 0$, $B' \in G'_{\delta'}$. Writing $C := \Phi^{-1}(B')$ and $D := \Phi^{-1}((B')_{\kappa'})$ for short, $\mu(C) = \mu'(B') \neq 0$ and $\text{diam}(D) \leq L \cdot \text{diam}((B')_{\kappa'}) = L\sqrt{d}(1+2\kappa')\delta'$. For every $\delta > 0$, the choice of which is postponed at the moment, the $\delta$-boxes constitute a covering of $\mathbb{R}^d$. Hence there must be one of them, say $B_C$, which meets $C$ and is no $\mu$-nullset. Consequently $B_C$ is in $G_\delta$. Choosing $\delta$ suitable can result in $(B_C)_\kappa \subset D$ or $D \subset (B_C)_\kappa$, as desired.

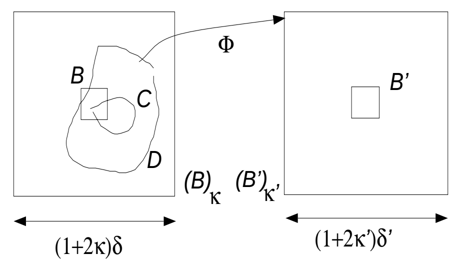


Figure 1.4: $\Phi$ *and* $\Phi^{-1}$ *are lipschitzian.*

**a)** $q \geq 0$**:** The constructed box $B_C$ should be large. The claim to prove is: with $\eta_1 := \kappa^{-1}L\sqrt{d}(1+2\kappa')$ and $b_1 := (L\sqrt{d}\upsilon^{-1}\eta_1 + 2)^d$ the estimate

$$S'_{\delta'}(q,\kappa') = \sum_{B'\in G'_{\delta'}} \mu'((B')_{\kappa'})^q \leq b_1 \sum_{B\in G_\delta} \mu((B)_\kappa)^q = b_1 S_\delta(q,\kappa) \tag{1.2}$$

is valid for every $\delta \in [\eta_1\delta', \upsilon^{-1}\eta_1\delta']$. First, the definition of $\eta_1$ guarantees, that $\kappa\delta \geq \text{diam}(D)$. Thus $D$ is contained in $(B_C)_\kappa$ (see Fig. 1.4), and

$$\mu((B_C)_\kappa) \geq \mu(D) = \mu'((B')_{\kappa'}) \neq 0.$$

The same estimate holds for the $q$–th powers and the larger term in $S_\delta(q,\kappa)$ is found. The given construction is not one-to-one, but the number of all $\delta'$-boxes $B'$ for which the same box $B_C$ has been constructed, is bounded by the constant $b_1$. Repeating each term in $S_\delta(q,\kappa)$ $b_1$ times produces a counterpart for every term in $S'_{\delta'}(q,\kappa')$. This establishes (1.2). To prove the mentioned property of $b_1$ fix $B$. If $B$ has been constructed as the counterpart of some $B'$, then $\Phi(B)\cap B' \neq \emptyset$. But since $\operatorname{diam}(\Phi(B)) \le L\sqrt{d}\delta \le L\sqrt{d}\upsilon^{-1}\eta_1\delta'$, there are at the most $b_1$ boxes $B'$ which can intersect $\Phi(B)$. So, we are done.

**b)** $q<0$**:** This time the constructed box $B$ should be small. For any $\delta \le \eta_2\cdot\delta' := ((1+\kappa)\sqrt{d}L)^{-1}\kappa'\delta'$, the set $(B)_\kappa$ is contained in $D$ and so $0\neq\mu(B)\le\mu((B)_\kappa)\le\mu(D)=\mu'((B')_{\kappa'})$. Rising the inequality to the $q$–th power reverses the sign. Similar as in a) the number of boxes $B'$, which lead to the same $B$, is bounded: since $\operatorname{diam}(\Phi(B))\le L\sqrt{d}\eta_2\delta'$, at the most $b_2=(L\sqrt{d}\eta_2+2)^d$ $\delta'$-boxes can meet $\Phi(B)$. This implies immediately:

$$S'_{\delta'}(q,\kappa')\le b_2 S_\delta(q,\kappa) \qquad \forall\,\delta\le\eta_2\delta'. \tag{1.3}$$

**ii)** Interchanging $K$ with $K'$ and $\kappa$ with $\kappa'$ yields:

$$\begin{array}{llll} \textbf{a)}\ q\ge 0: & S_\delta(q,\kappa) \le b_3 S'_{\delta'}(q,\kappa') & \forall\,\delta'\in[\eta_3\delta,\upsilon^{-1}\eta_3\delta]. & (1.4)\\ \textbf{b)}\ q<0: & S_\delta(q,\kappa) \le b_4 S'_{\delta'}(q,\kappa') & \forall\,\delta'\le\eta_4\delta. & (1.5)\end{array}$$

**iii)** It is easy to derive the desired conclusions from i) and ii). The proof is only given for the limes superior:

**a)** $q\ge 0$**:** Applying (1.4) with $\delta'=\eta_3\delta$ implies:

$$\frac{\log S_\delta(q,\kappa)}{-\log\delta}\le\frac{\log S'_{\delta'}(q,\kappa')+\log b_3}{-\log(\delta')+\log\eta_3},$$

hence

$$\limsup_{\delta\downarrow 0}\frac{\log S_\delta(q,\kappa)}{-\log\delta}\le\limsup_{\delta'\downarrow 0}\frac{\log S'_{\delta'}(q,\kappa')}{-\log\delta'}.$$

On the other hand, given $\delta'$, $n$ can be chosen such that $\delta_n\ge\eta_1\delta'>\delta_{n+1}$. Then $\delta_n\le\upsilon^{-1}\delta_{n+1}\le\eta_1\upsilon^{-1}\delta'$, and with (1.2)

$$\frac{\log S'_{\delta'}(q,\kappa')}{-\log\delta'}\le\frac{\log S_{\delta_n}(q,\kappa)+\log b_1}{-\log(\delta_n)+\log\eta_1},$$

thus

$$\limsup_{\delta'\downarrow 0}\frac{\log S'_{\delta'}(q,\kappa')}{-\log\delta'}\le\limsup_{n\to\infty}\frac{\log S_{\delta_n}(q,\kappa)}{-\log\delta_n}\le\limsup_{\delta\downarrow 0}\frac{\log S_\delta(q,\kappa)}{-\log\delta}.$$



**b)** $q<0$**:** The argument is the same as in a) except that $n$ is such that $\delta_{n-1}\ge\eta_2\delta'>\delta_n$. Applying then (1.3) to $\delta=\delta_n$ gives the estimate of the numerator, and $\delta_n\ge\upsilon\delta_{n-1}\ge\upsilon\eta_2\delta'$ bounds the denominator.

Summarizing:

$$\limsup_{\delta\downarrow 0}\frac{\log S'_\delta(q,\kappa')}{-\log\delta}=\limsup_{\delta\downarrow 0}\frac{\log S_\delta(q,\kappa)}{-\log\delta}=\limsup_{n\to\infty}\frac{\log S_{\delta_n}(q,\kappa)}{-\log\delta_n}\qquad\forall q\in\mathbb{R}.$$

Thereby it is of course possible to choose $\kappa'=1$ or $\Phi$ to be the identity. ◇

It is interesting to know how the modification in the measurement of the concentrations of $\mu$ affects the singularity exponents.

**Proposition 1.6** *Let $(\delta_n)_{n\in\mathbb{N}}$ be an admissible sequence. Then*

$$T(q)=\tau(q)=\limsup_{n\to\infty}\frac{\log s_{\delta_n}(q)}{-\log\delta_n},\qquad \underline{T}(q)=\underline{\tau}(q)=\liminf_{n\to\infty}\frac{\log s_{\delta_n}(q)}{-\log\delta_n}$$

*for any $q\ge 0$ and*

$$T(q)\le\tau(q),\qquad \underline{T}(q)\le\underline{\tau}(q)$$

*for any $q<0$.*

**Remark** Example 1.1 provides a multifractal and two admissible sequences $(\delta_n)_{n\in\mathbb{N}}$ and $({\delta_n}^*)_{n\in\mathbb{N}}$ with

$$T(q)=\lim_{n\to\infty}\frac{\log s_{\delta_n^*}(q)}{-\log\delta_n^*}<\lim_{n\to\infty}\frac{\log s_{\delta_n}(q)}{-\log\delta_n}=\tau(q)=\infty\qquad\forall q<0.$$

**Proof** The argument is only given for $T(q)$ und $\tau(q)$.
The first part is easy: the measure of each box $B\in G_\delta$ is nonvanishing and certainly smaller than the one of $(B)_1$. Since the sums $s_\delta(q)$ und $S_\delta(q)$ run over the same boxes, this gives immediately

$$\tau(q)\le T(q)\quad\forall q\ge 0\qquad\qquad \tau(q)\ge T(q)\quad\forall q<0. \tag{1.6}$$

Now take $q\ge 0$. To every term in $S_\delta(q)$ a greater one will be found in $s_\delta(q)$. Take $B\in G_\delta$. There are exactly $b_5=3^d$ $\delta$-boxes $C$ intersecting the parallelbody $(B)_1$. Letting $C_B$ to be one of maximal measure among them yields

$$0<\mu((B)_1)\le\sum_{C\cap(B)_1\neq\emptyset}\mu(C)\le b_5\mu(C_B). \tag{1.7}$$

Now fix a $\delta$-box $C$ and ask, how many $\delta$-boxes $B$ could possibly share $C$ as $C_B$. Since $C$ und $(B)_1$ meet, so do $B$ and $(C)_1$. But $(C)_1$ intersects at the most $b_5$ $\delta$-boxes $B$. So, observing (1.7) and repeating the terms in $s_\delta(q)$ results in:

$$S_\delta(q)\le b_5^q\sum_{B\in G_\delta}\mu(C_B)^q\le b_5^q\cdot b_5\sum_{C\in G_\delta}\mu(C)^q=b_5^{q+1}s_\delta(q)\qquad\forall\delta>0.$$

Applying this to the admissible sequence $\delta_n$, proposition 1.5 and (1.6) yield:

$$T(q) = \limsup_{n\to\infty} \frac{\log S_{\delta_n}(q)}{-\log \delta_n} \leq \limsup_{n\to\infty} \frac{\log s_{\delta_n}(q)}{-\log \delta_n} \leq \tau(q) \leq T(q).$$

It is not possible to carry out this construction for negative $q$. Of course to every $\delta$-box $C$ one may find a $\delta'$-box $B$, the parallelbody of which is contained in $C$. But as example 1.1 shows, it is not possible to guarantee simultaneously $\mu(B) \neq 0$ and $\delta' \geq \text{const} \cdot \delta$. ◇

The definition of the spectrum will now be modified in the same way as the one for the singularity exponents.

**Definition 1.7 (Spectrum)** *The* semispectra $F^+$ *and* $F^-$ *are given through*

$$N_\delta(\alpha) := \#\{B \in G_\delta \,:\, \mu((B)_1) \geq \delta^\alpha\} \qquad M_\delta(\alpha) := \#\{B \in G_\delta \,:\, \mu((B)_1) < \delta^\alpha\}$$

$$F^+(\alpha) := \limsup_{\delta\downarrow 0} \frac{\log N_\delta(\alpha)}{-\log \delta} \qquad F^-(\alpha) := \limsup_{\delta\downarrow 0} \frac{\log M_\delta(\alpha)}{-\log \delta}.$$

*They have to be considered as auxiliary functions enabling a better treatment of the* spectrum, *which is defined as follows:*

$$F(\alpha) := \lim_{\varepsilon\downarrow 0} \limsup_{\delta\downarrow 0} \frac{\log(N_\delta(\alpha+\varepsilon) - N_\delta(\alpha-\varepsilon))}{-\log \delta}.$$

*Thereby the value $-\infty$ is allowed.*

It should be emphasized that this notion uses measures $\mu((B)_1)$ where the boxes $B$ have been selected by the condition $\mu(B) \neq 0$. In order to discuss problems of convergence $F(\alpha)$ will be called *grid-regular*, whenever the $\limsup_{\delta\to 0}$ is actually a limes, i.e. whenever

$$F(\alpha) = \lim_{\varepsilon\downarrow 0} \lim_{\delta\downarrow 0} \frac{\log(N_\delta(\alpha+\varepsilon) - N_\delta(\alpha-\varepsilon))}{-\log \delta} \tag{1.8}$$

for a particular $\alpha$.

$F$ will prove to be free of the kind of anomalous behaviour $f$ suffers from. For instance, for the middle third Cantor measure (Ex. 1.1) one finds $F \equiv f^*$, while $f$ is only known in the increasing part.

First the regularities of the semispectra corresponding to the ones of the singularity exponents will be proven: The equality $T = \tau$ $(q \geq 0)$ translates to $F^+(\alpha+) = f^+(\alpha+)$. In words: the rising part of the spectrum is essentially left unaffected by the replacement of $\mu(B)$ by $\mu((B)_1)$. Moreover, $F$ is essentially invariant under admissible coordinate transformations and one may use an admissible sequence for its calculation.

The functions $F^+(\alpha)$ and $f^+(\alpha)$ are monotonous increasing, $F^-(\alpha)$ and $f^-(\alpha)$ monotonous decreasing. Thus the onesided limites $F^+(\alpha+) = \lim_{\varepsilon\downarrow 0} F^+(\alpha+\varepsilon)$, etc. exist.



**Proposition 1.8** *The values $F^+(\alpha+)$, $F^+(\alpha-)$, $F^-(\alpha+)$ and $F^-(\alpha-)$ are invariant under admissible coordinate transformations. Moreover, they can be calculated through admissible sequences and do not depend on the particular choice $\kappa = 1$ of the constant factor the boxes are enlarged with. In addition*

$$F^+(\alpha\pm) = f^+(\alpha\pm), \qquad F^-(\alpha\pm) \leq f^-(\alpha\pm).$$

**Remark** Equality between $F^-$ and $f^-$ may fail as example 2.1 shows.

**Proof** The labeling of the steps in this proof as well as some constants correspond to the ones in the proof of proposition 1.5, since the basic idea of the argumentation is the same. As above the notation $\mu' = \Phi_*\mu$, $K' = \Phi(K)$ etc. is used. The counting function $N_\delta(\alpha, \kappa)$ corresponding to $\mu$ and enlarged boxes $(B)_\kappa$ is compared with $N'_{\delta'}(\alpha, \kappa')$ corresponding to $\mu'$ and enlargement $\kappa'$. This is done to take advantage of symmetries.

**o)** The case $\alpha < 0$ is trivial:

- **a)** $N'_{\delta'}(\alpha, \kappa') = N_\delta(\alpha, \kappa) = n_\delta(\alpha) = 0$, $F^+(\alpha) = f^+(\alpha) = -\infty$.
- **b)** $M_\delta(\alpha, \kappa) = m_\delta(\alpha) = \#G_\delta = S_\delta(0, \kappa)$ and $M'_{\delta'}(\alpha, \kappa') = m'_{\delta'}(\alpha, \kappa') = \#G'_{\delta'} = S'_{\delta'}(0, \kappa')$. Furthermore, $F^-(\alpha) = f^-(\alpha) = T(0)$ due to (1.2) and (1.4) (with $q = 0$).

For the remainder $\alpha \geq 0$ is assumed.

**i)** Take $\delta' > 0$, $B' \in G^{\mu'}_{\delta'}$ and $\varepsilon > 0$ arbitrarily. Since $\Phi^{-1}(B')$ has positive $\mu$-measure, there must be a box $B$ from $G^\mu_\delta$ intersecting it. The intention is to compare $\mu'((B')_{\kappa'})$ with $\mu((B)_\kappa)$ for a suitable $\delta$.

**a) Concerning $N_\delta$:** Take $B'$ with $\mu'((B')_{\kappa'}) \geq (\delta')^\alpha$ and choose $\delta$ such that $(B)_\kappa$ has large measure too: for $\delta \in [\eta_1\delta', \upsilon^{-1}\eta_1\delta']$ the set $\Phi^{-1}((B')_{\kappa'})$ is contained in $(B)_\kappa$. For $\delta$ small enough, i.e. $\delta \leq (\upsilon/\eta_1)^{\alpha/\varepsilon}$, one obtains

$$\mu((B)_\kappa) \geq \mu'((B')_{\kappa'}) \geq (\delta')^\alpha \geq \upsilon^\alpha \eta_1^{-\alpha} \delta^\alpha \geq \delta^{\alpha+\varepsilon}.$$

So far a $B \in G^\mu_\delta$ was constructed for every $B' \in G^{\mu'}_{\delta'}$. Several $B'$ can lead to the same $B$. But at the most $b_1$ such $B'$ can intersect the same (fixed) $B$. Consequently

$$N'_{\delta'}(\alpha, \kappa') \leq b_1 N_\delta(\alpha+\varepsilon, \kappa).$$

Moreover, for sufficiently small $\delta' > 0$ there is an integer $n$ such that $\delta_n \geq \eta_1\delta' > \delta_{n+1}$. This implies $\delta_n \leq \upsilon^{-1}\delta_{n+1} \leq \upsilon^{-1}\eta_1\delta'$, allowing to conclude from the considerations above ($\delta = \delta_n$, $\kappa = 1$):

$$\limsup_{\delta'\downarrow 0} \frac{\log N'_{\delta'}(\alpha, \kappa')}{-\log \delta'} \leq \limsup_{n\to\infty} \frac{\log N_{\delta_n}(\alpha+\varepsilon)}{-\log \delta_n} \leq F^+(\alpha+\varepsilon). \tag{1.9}$$

**b) Concerning $M_\delta$:** Take $B'$ with $\mu'((B')_{\kappa'}) < (\delta')^\alpha$ and choose $\delta$ such that $(B)_\kappa$ has small measure too: for $\delta \leq \eta_2\delta'$ the set $(B)_\kappa$ is contained in $\Phi^{-1}((B')_{\kappa'})$. So, if $\delta \leq {\eta_2}^{\alpha/\varepsilon}$ then

$$\mu((B)_\kappa) \leq \mu'((B')_{\kappa'}) < (\delta')^\alpha \leq \eta_2^{-\alpha}\delta^\alpha \leq \delta^{\alpha-\varepsilon}.$$

And by a similar argument as above

$$M'_{\delta'}(\alpha, \kappa') \leq b_2 M_\delta(\alpha - \varepsilon, \kappa).$$

Moreover, if $n \in \mathbb{N}$ is such that $\delta_{n-1} \geq \eta_2\delta' > \delta_n$ then $\delta_n \geq \upsilon\delta_{n-1} \geq \upsilon\eta_2\delta'$. Using this to estimate the denominator the considerations above yield ($\delta = \delta_n$, $\kappa = 1$):

$$\limsup_{\delta'\downarrow 0} \frac{\log M'_{\delta'}(\alpha, \kappa')}{-\log \delta'} \leq \limsup_{n\to\infty} \frac{\log M_{\delta_n}(\alpha - \varepsilon)}{-\log \delta_n} \leq F^-(\alpha - \varepsilon). \quad (1.10)$$

**ii)** Interchanging $K$ with $K'$ and $\kappa$ with $\kappa'$ in i) gives:

**a) Concerning $N_\delta$:** $N_\delta(\alpha, \kappa) \leq b_3 N'_{\delta'}(\alpha + \varepsilon, \kappa')$ for $\delta' \in [\eta_3\delta, \upsilon^{-1}\eta_3\delta]$ and $\delta$ sufficiently small. With $\kappa = 1$ and $\delta' = \eta_3\delta$

$$F^+(\alpha) \leq \limsup_{\delta'\downarrow 0} \frac{\log N'_{\delta'}(\alpha + \varepsilon, \kappa')}{-\log \delta'}, \quad (1.11)$$

and with $\kappa = \kappa' = 1$, $\Phi$=identity

$$\liminf_{n\to\infty} \frac{\log N_{\delta_n}(\alpha)}{-\log \delta_n} \leq \liminf_{\delta'\downarrow 0} \frac{\log N_{\delta'}(\alpha + \varepsilon)}{-\log \delta'}. \quad (1.12)$$

Thereby one has to pick the one integer $n$ satisfying $\upsilon^{-1}\eta_3\delta_{n+1} < \delta' \leq \upsilon^{-1}\eta_3\delta_n$ for given $\delta'$. (1.12) will be used later.

**b) Concerning $M_\delta$:** $M_\delta(\alpha, \kappa) \leq b_4 M'_{\delta'}(\alpha - \varepsilon, \kappa')$ for $\delta' = \eta_4\delta$ and $\delta$ sufficiently small. With $\kappa = 1$:

$$F^-(\alpha) \leq \limsup_{\delta'\downarrow 0} \frac{\log M'_{\delta'}(\alpha - \varepsilon, \kappa')}{-\log \delta'}. \quad (1.13)$$

**iii)** Applying (1.9) and (1.11), resp. (1.10) and (1.13) with different values $\alpha$, $\alpha+\varepsilon$, $\alpha - \varepsilon$ and so on, the invariance of $F^+(\alpha\pm)$ resp. $F^-(\alpha\pm)$ is readily derived.

Finally let us compare $F$ with the former formalism $f$.

**iv)** The trivial estimate $\mu(B) \leq \mu((B)_1)$ implies $f^+(\alpha) \leq F^+(\alpha)$ and $f^-(\alpha) \geq F^-(\alpha)$.

**v)** Let $\varepsilon > 0$. To every given $\delta$-box $B$ with $\mu((B)_1) \geq \delta^\alpha$ (1.7) gives a $\delta$-box $C_B$ with

$$\mu(C_B) \geq b_5^{-1}\mu((B)_1) \geq \delta^{\alpha+\varepsilon},$$

provided $\delta \leq {b_5^{-1/\varepsilon}}$. At the most $b_5$ boxes $B$ may generate the same (fixed) $C_B$. Consequently, $N_\delta(\alpha) \leq b_5 \cdot n_\delta(\alpha + \varepsilon)$ and $F^+(\alpha) \leq f^+(\alpha + \varepsilon)$.



**vi)** From iv) and v) $f^+(\alpha+) = F^+(\alpha+)$ and $f^+(\alpha-) = F^+(\alpha-)$. ◇

Besides being monotonous, $F^+$ and $F^-$ possess further simple properties. In the region $\alpha < 0$ for instance $F^+(\alpha) = -\infty$ und $F^-(\alpha) = T(0)$. This is generalized by:

**Lemma 1.9**

$$\underline{T}(0) \leq \max(F^+(\alpha), F^-(\alpha)) \leq T(0) \qquad \forall \alpha \in \mathbb{R}.$$

**Proof** The upper bound is trivial due to $\#G_\delta = S_\delta(0)$. Assume now $F^+(\alpha) < T(0)$. Choose $\eta > 0$ such, that $F^+(\alpha) + 3\eta \leq T(0)$. Then choose a sequence $(\delta_n)_{n\in\mathbb{N}}$ tending to zero such that

$$S_{\delta_n}(0) \geq {\delta_n}^{-T(0)+\eta} \quad \text{and} \quad {\delta_n}^\eta \leq \frac{1}{2} \quad \forall n \in \mathbb{N}.$$

Now there is $n_0(\eta)$ such that

$$N_{\delta_n}(\alpha) \leq {\delta_n}^{-F^+(\alpha)-\eta} \leq {\delta_n}^\eta {\delta_n}^{-T(0)+\eta} \leq \frac{1}{2} S_{\delta_n}(0) \quad \forall n \geq n_0.$$

This implies

$$M_{\delta_n}(\alpha) = S_{\delta_n}(0) - N_{\delta_n}(\alpha) \geq \frac{1}{2} S_{\delta_n}(0)$$

and thus $F^-(\alpha) \geq \underline{T}(0)$, which proves the claim. ◇

Finally the relation between the semispectra and the spectrum is investigated. Certainly

$$F(\alpha) \leq \min(F^+(\alpha+), F^-(\alpha-)). \quad (1.14)$$

The intuition, that equality in (1.14) should hold, is supported by lemma 1.9, by the monotonicity of the semispectra and the intuition that $F(\alpha) = \sup\{F(t) : t < \alpha\} = F^+(\alpha)$ in the increasing part and $F(\alpha) = \sup\{F(t) : t > \alpha\} = F^-(\alpha)$ in the decreasing part. Consider figure 1.5, which suggests for which $\alpha$ one can expect $F(\alpha) = F^+(\alpha)$: at the points where $F^+(\alpha)$ is either strictly monotonous or equal to $-\infty$. In fact, to prove the desired equality it is sufficient to require an even weaker condition: A function $t$ is called *quasi increasing at* $\alpha$, if

$$\alpha' < \alpha < \alpha'' \qquad \text{implies} \qquad t(\alpha') < t(\alpha'').$$

Thereby $-\infty < -\infty$ by definition. When the same holds with reversed inequality, $t$ is called *quasi decreasing at* $\alpha$.

**Proposition 1.10 a)** *If $F^+$ is quasi increasing at $\alpha$, then*

$$F(\alpha) = F^+(\alpha+) = f(\alpha)$$

*and $F(\alpha)$ shares the properties of $F^+(\alpha+)$ stated in proposition 1.8. If in addition for any sufficiently small $\varepsilon > 0$ there is an admissible sequence $\delta_n$ for which*

$$\lim_{n\to\infty} \frac{\log N_{\delta_n}(\alpha + \varepsilon)}{-\log \delta_n}$$

*exists, then $F(\alpha)$ is grid-regular.*

**b)** *If $F^-$ is quasi decreasing at $\alpha$, then*

$$F(\alpha) = F^-(\alpha-)$$

*and $F(\alpha)$ shares the properties of $F^-(\alpha-)$ stated in proposition 1.8. If in addition for any sufficiently small $\varepsilon > 0$ there is an admissible sequence $\delta_n$ for which*

$$\lim_{n\to\infty} \frac{\log M_{\delta_n}(\alpha-\varepsilon)}{-\log \delta_n}$$

*exists, then $F(\alpha)$ is grid-regular.*

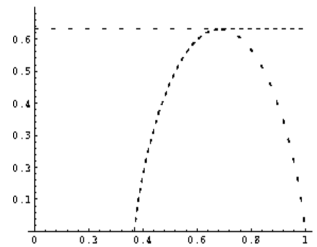


Figure 1.5: *The connection between semispectra and spectrum illustrated using the spectrum of the middle third Cantor set: the increasing part of $F$ is given by $F^+$, the decreasing part by $F^-$. Thus $F = \min(F^+, F^-)$ in this example.*

**Proof** For reasons of simplicity only the case $\kappa = 1$ is considered. However, it is immediate that the arguments below apply to any choice of $\kappa$. From this follows the independence of $F$ from $\kappa$ under the stated hypothesis.

**i)** Assume that $\varepsilon$ satisfies $F^+(\alpha-\varepsilon) < F^+(\alpha+\varepsilon)$. Given the case $F^+(\alpha-\varepsilon) \neq -\infty$ this means that $N_\delta(\alpha+\varepsilon)$ grows essentially faster than $N_\delta(\alpha-\varepsilon)$. More precisely: choose $\eta > 0$ such that $F^+(\alpha-\varepsilon) + 3\eta \leq F^+(\alpha+\varepsilon)$ and take a sequence $(\delta_n)_{n\in\mathbb{N}}$ which gives the lim sup $F^+(\alpha+\varepsilon)$. This means that there is $n_0(\eta)$ with

$$N_{\delta_n}(\alpha+\varepsilon) \geq {\delta_n}^{-F^+(\alpha+\varepsilon)+\eta}, \quad N_{\delta_n}(\alpha-\varepsilon) \leq {\delta_n}^{-F^+(\alpha-\varepsilon)-\eta}, \quad {\delta_n}^{\eta} \leq \frac{1}{2} \tag{1.15}$$

for all $n \geq n_0$. This implies

$$\frac{1}{2} N_{\delta_n}(\alpha+\varepsilon) \leq N_{\delta_n}(\alpha+\varepsilon) - N_{\delta_n}(\alpha-\varepsilon) \qquad \forall n \geq n_0. \tag{1.16}$$

Observing $N_\delta(\alpha-\varepsilon) \geq 0$ gives

$$F^+(\alpha+\varepsilon) \leq \limsup_{\delta\downarrow 0} \frac{\log\left(N_\delta(\alpha+\varepsilon) - N_\delta(\alpha-\varepsilon)\right)}{-\log\delta} \leq F^+(\alpha+\varepsilon) \tag{1.17}$$



Given the case $F^+(\alpha-\varepsilon) = -\infty$ there is a $\delta_0 > 0$ with $N_\delta(\alpha-\varepsilon) = 0 \; \forall \delta \leq \delta_0$. The inequality (1.17) holds then trivially.

**ii)** Provided $F^+(\alpha)$ or $F^+(\alpha+)$ is quasi increasing at $\alpha$, (1.17) holds for all $\varepsilon > 0$, implying $F(\alpha) = F^+(\alpha+)$. If, on the other hand, $F^-$ is quasi decreasing at $\alpha$, the term $M_\delta(\alpha-\varepsilon)$ dominates the difference $M_\delta(\alpha-\varepsilon) - M_\delta(\alpha+\varepsilon)$, which equals $N_\delta(\alpha+\varepsilon) - N_\delta(\alpha-\varepsilon)$. Hence $F(\alpha) = F^-(\alpha-)$.

**iii)** Concerning the grid-regularity in a): by (1.12)

$$\liminf_{\delta\downarrow 0} \frac{\log N_\delta(\alpha+\varepsilon)}{-\log\delta} \geq \liminf_{n\to\infty} \frac{\log N_{\delta_n}(\alpha+\varepsilon/2)}{-\log\delta_n} = F^+(\alpha+\varepsilon/2) > F^+(\alpha-\varepsilon)$$

for sufficiently small $\varepsilon > 0$. Take first the case $F^+(\alpha-\varepsilon) \neq -\infty$. With $\eta > 0$ satisfying $F^+(\alpha-\varepsilon) + 3\eta \leq F^+(\alpha+\varepsilon/2)$ it may be proceeded as in i) finding (1.16) to hold not only for a particular sequence but for all sufficiently small $\delta > 0$. As a consequence

$$\lim_{\delta\downarrow 0} \frac{\log\left(N_\delta(\alpha+\varepsilon) - N_\delta(\alpha-\varepsilon)\right)}{-\log\delta} = \lim_{\delta\downarrow 0} \frac{\log N_\delta(\alpha+\varepsilon)}{-\log\delta} = F^+(\alpha+\varepsilon),$$

which is trivial in the case $F^+(\alpha-\varepsilon) = -\infty$. So (1.8) exists and takes the value $F^+(\alpha+) = F(\alpha)$ by ii). In the corresponding situation in b) one finds the existence of (1.8) and its value $F^-(\alpha-) = F(\alpha)$. $\diamond$

Finally the modified definition of the generalized dimensions reads as:

**Definition 1.11 (Generalized Dimensions)**

$$D_q := \frac{1}{1-q} T(q) \quad (q \neq 1) \quad D_1 := \limsup_{\delta\downarrow 0} \frac{1}{\log(\delta)} \frac{\sum_{B\in G_\delta} \mu((B)_1)\log\mu((B)_1)}{\sum_{B\in G_\delta} \mu((B)_1)}.$$

## 1.3 The Legendre Transform of the Spectrum

As it was already mentioned—and what will be proved in this section—applying the transformation of Legendre to $F$ leads to $T$. So, once the spectrum is computed the singularity exponents are readily obtained.

In typical applications, however, one will meet the converse situation: one would like to be able to deduce the spectrum from the singularity exponents. This matter would be straightforward if differentiability and concavity of the spectrum could be assumed: $F$ would simply be the transform of $T$. But proofs establishing such qualities [BR, Lan] work with a multifractal notion 'tailored to multiplicative cascades' and do not apply to our formalism. Moreover, we put forward counterexamples with nonconcave spectrum $F$ (Ex. 2.15 and 2.16). For these multifractals the singularity

exponents contain less information than the spectrum and cannot be used to obtain the entire $F(\alpha)$-curve.
Calculating $F$ as the Legendre transform of $T$ is thus valid only if it is known *in advance* that the spectrum is differentiable and concave. It is more convenient to start with the semispectra $F^+$ and $F^-$, which are *a priori* known to be monotonous. This property will turn out to be enough to prove that $T$ determines the semispectra and even the spectrum under reasonable conditions. In particular, no differentiability has to be assumed about $F^\pm$. To make the terminology precise we define:

**Definition 1.12 (Convexity)** *A real-valued function $t$ with domain $\mathbb{D}$ is called* convex at the point $\alpha_0$*, iff there are real numbers $q$ and $r$ such that*

$$r + q\alpha \leq t(\alpha) \qquad \forall \alpha \in \mathbb{D} \tag{1.18}$$

*with equality holding for $\alpha = \alpha_0$. We call it* strictly convex at $\alpha_0$ *iff the inequality (1.18) is strict for all $\alpha \neq \alpha_0$ in $\mathbb{D}$. If $t$ is convex at all points of $\mathbb{D}$, we call it* convex in $\mathbb{D}$*. If (1.18) holds with reversed sign we call $t$* concave*, respectively* strictly concave.

First a simple result on the Legendre transform is needed. Instead of condemning the interested reader to an exhausting search of this vast and well known topic we feel that the proofs are short enough to be presented right here. The assumptions are tailored to concave functions $t$ and suit our purpose.

**Lemma 1.13** *Let $t : \mathbb{R} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ be arbitrary and define its* Legendre transform *by $l(q) := \sup_{\alpha\in\mathbb{R}}(t(\alpha) - q\alpha)$ for $q \in \mathbb{R}$. Then, either $l$ and $t$ are both identically $-\infty$ or there exist four numbers $A_1 \leq A_2 \leq A_3 \leq A_4$ in $\overline{\mathbb{R}}$ such that*

**a)** *$l(q)$ equals $\infty$ for $q$ not in $[A_1, A_4]$,*

**b)** *$l$ is real-valued, continuous and convex in $\mathbb{D} := ]A_1, A_4[$,*

**c)** *$l$ is strictly monotonous decreasing in $]A_1, A_2[$,*

**d)** *$l$ is constant and takes its minimal value in $[A_2, A_3]$,*

**e)** *$l$ is strictly monotonous increasing in $]A_3, A_4[$.*

**f)** (Touching point of $t$) *For all $q$ in $]A_1, A_4[$ there is a real number $x(q)$—strictly positive for $q \in ]A_1, A_2[$ and strictly negative for $q \in ]A_3, A_4[$—and a sequence $(x_n)_{n\in\mathbb{N}}$ such that*

$$x_n \to x(q) \qquad t(x_n) \to l(q) + qx(q) \quad (n \to \infty).$$

*Moreover, if $l$ is differentiable at $q$, then $x(q) = -l'(q)$ is the only touching point.*



**g)** *If $l$ is linear in a neighbourhood of $q$, then $t$ is not differentiable at $\alpha = -l'(q)$.*

**h)** *If $A_1 < A_4$ and $l(A_1)$ resp. $l(A_4)$ is finite, then $l(A_1) = l(A_1+)$ resp. $l(A_4) = l(A_4-)$.*

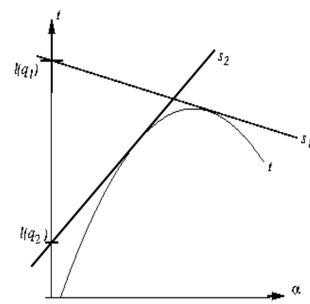


Figure 1.6: *The Legendre transform*

**Proof**

**i)** If there is a $q$ with $l(q) = -\infty$, then $t$ must be identically $-\infty$ and hence $l$ as well.

**ii)** Let us assume that $q_1 < q_2$ are such that $l(q_1)$ and $l(q_2)$ are real numbers. Then, $t$ is bounded from above by the two linear functions $s_k(\alpha) := l(q_k) + \alpha q_k$ (k=1,2). These two meet at $\alpha_{12} = -(l(q_1) - l(q_2))(q_1 - q_2)^{-1}$. The linear function with slope $q_3 \in ]q_1, q_2[$ which passes through the intersection point $(\alpha_{12}, s_1(\alpha_{12}))$ bounds $t$ as well and hence $l(q_3)$ must be a real number too. From this a) and the first part of b) result.

**iii)** Continuity as well as c), d) and e) follow directly from convexity. To prove the latter the existence of $x(q)$ is required first. Let $q_3 \in ]q_1, q_2[$ as above. By definition there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $t(x_n) - q_3x_n \to l(q_3)$. Since $s_1$ and $s_2$ bound $t$, the accumulation points of $(x_n)_{n\in\mathbb{N}}$ must all lie between $\alpha_{13}$ and $\alpha_{23}$. Choose a converging subsequence and call it again $x_n$. Since $x_n$ converges say to $x(q_3)$, $t(x_n)$ must converge to $r := s_3(x(q_3))$. The explicit formula of $\alpha_{12}$ now gives f), as soon as c) – e) are proven, and g) follows from f).

**iv)** Next, the convexity of $l$ is shown using only the existence of $x(q)$. First, $l(q_3) = r - q_3x(q_3)$ by definition of $r$. Letting $n \to \infty$, $l(q) \geq t(x_n) - qx_n$ gives $l(q) \geq r - qx(q_3)$ for all $q$. Hence $l$ is convex at $q_3$.

**v)** It remains to show h). Take $q_0 \in \{A_1, A_4\}$. There is still a sequence $x_n$ as above which may be assumed to converge to $x(q_0) \in \overline{\mathbb{R}}$. If $x(q_0)$ is finite the onesided continuity follows from convexity. If $x_n \to \infty$, then for any $\varepsilon > 0$ and $q \in \mathbb{R}$ there is $n_0 \in \mathbb{N}$ with $l(q) \geq t(x_n) - qx_n \geq l(q_0) - \varepsilon + (q_0 - q)x_n$ for $n \geq n_0$. For $q < q_0$ one concludes $l(q) = \infty$, hence $q_0 = A_1$. Letting $q \downarrow q_0$ yields $l(q_0+) \geq l(q_0) - \varepsilon$. On the other hand, $l(q_0) \geq l(q_0+)$ since by iv) $l$ is even with domain $[A_1, A_4[$ convex at any $q \in ]A_1, A_4[$. This proves the claim. The case $x_n \to -\infty$ can be treated in a similar manner. ◇

Next, the Legendre transforms of the semispectra are calculated, which turn out to equal $T$ in the essential parts. Since $F^+$ is increasing and $F^-$ decreasing, one may only expect $T$ to equal the Legendre transform of $F^+$ for $q > 0$ and the one of $F^-$ for $q < 0$. Moreover, there is an inherent lack of information about $T(0)$. So it is useful to express $T$ as the transform of one single function for all $q$. To this end we set

**Definition 1.14**

$$F^m(\alpha) := \min\left(F^+(\alpha+), F^-(\alpha-)\right) \quad \text{and} \quad \alpha^* := \sup\{\alpha \in \mathbb{R} : F^+(\alpha) < F^-(\alpha)\}.$$

**Proposition 1.15**

$$T(q) = \sup_{\alpha\in\mathbb{R}}(F^+(\alpha) - q\alpha) = \sup_{\alpha\in\mathbb{R}}(F^m(\alpha) - q\alpha) \qquad \forall q > 0,$$

*and*

$$T(q) = \sup_{\alpha\in\mathbb{R}}(F^-(\alpha) - q\alpha) = \sup_{\alpha\in\mathbb{R}}(F^m(\alpha) - q\alpha) \qquad \forall q < 0.$$

*Moreover, if $\alpha^* < \infty$, then*

$$T(0) = \sup_{\alpha\in\mathbb{R}} F^m(\alpha) = \sup_{\alpha\in\mathbb{R}} F^+(\alpha).$$

Example 2.14 provides a multifractal with $\alpha^* = \infty$ and $T(0) > \sup_{\alpha\in\mathbb{R}} F^+(\alpha)$.
**Remark** The proposition could as well be proven with $F$ replacing $F^m$. However, as theorem 1.1 shows, this gives no further information in our context.
**Proof** The idea of the first part is borrowed from J. K. Falconer. We will constantly make use of the boundedness and the monotonicity of the semispectra (Prop. 1.9). Write $l(q) = \sup_{\alpha\in\mathbb{R}}(F^m(\alpha) - q\alpha)$ and $L^\pm(q) = \sup_{\alpha\in\mathbb{R}}(F^\pm(\alpha) - q\alpha)$ for short.

**o)** As a consequence of the proposition to prove, $F^+$ cannot be identically $-\infty$ due to $T(1) = 0$. However, for this conclusion to hold the claim of the proposition has to be verified also for this case. So let us first treat the degenerate cases. First, consider $\alpha$ satisfying $F^+(\alpha) = -\infty$. Then, $\mu((B)_1) < \delta^\alpha$ for all $B \in G_\delta$, provided $\delta$ is small enough, and

$$S_\delta(1) \leq \#G_\delta \cdot \delta^\alpha \leq c \cdot \delta^{-d}\delta^\alpha,$$



where $c = (\mathrm{diam}(K) + 2)^d$ is a constant and $d$ is the dimension of $\mathbb{R}^d$. From this $0 = T(1) \leq d - \alpha$. In particular

$$\alpha > d \qquad \Rightarrow \qquad F^+(\alpha) \geq 0, \tag{1.19}$$

and $F^+ \equiv -\infty$ is impossible.
The second degenerate case is: $F^-(\alpha) \geq 0$ for all $\alpha$. Fix $q < 0$ and take $\alpha$ satisfying $F^-(\alpha) \geq 0$. Then, there are arbitrarily small $\delta$ such that $M_\delta(\alpha) \geq 1$. Thus, for these $\delta$ there is a box $B^* \in G_\delta$ with $\mu((B^*)_1) < \delta^\alpha$ and

$$S_\delta(q) \geq \mu((B^*)_1)^q \geq \delta^{q\alpha}.$$

From this $T(q) \geq -q\alpha$. By (1.19), the following implication holds:

$$F^-(\alpha) \geq 0 \;\forall\alpha \qquad \Rightarrow \qquad T(q) = \infty = L^-(q) = l(q) \quad (q < 0). \tag{1.20}$$

Having treated the degenerate cases $T$ shall now be estimated from below by $l$ and $L^\pm$.

**i)** Fix $q \geq 0$ and take $\alpha$ with $F^+(\alpha) > -\infty$. Of course $\alpha \geq 0$. For any $\gamma < F(\alpha)$ there are arbitrarily small $\delta > 0$ such that $\delta^{-\gamma} \leq N_\delta(\alpha)$. For such values $\delta$

$$S_\delta(q) = \sum_{B\in G_\delta} \mu((B)_1)^q \geq \sum_{\mu((B)_1)\geq\delta^\alpha} \mu((B)_1)^q \geq N_\delta(\alpha)\delta^{q\alpha} \geq \delta^{q\alpha-\gamma}$$

and hence

$$T(q) = \limsup_{\delta\downarrow 0} \frac{\log S_\delta(q)}{-\log\delta} \geq \gamma - q\alpha.$$

Since $\gamma$ is arbitrary, $T(q) \geq F^+(\alpha) - q\alpha$. This is trivial if $F^+(\alpha) = -\infty$. Thus $T(q) \geq L^+(q)$. To establish $T(q) \geq l(q)$ just take the limit $\alpha' \downarrow \alpha$ and obtain $T(q) \geq F^+(\alpha+) - q\alpha \geq F^m(\alpha) - q\alpha$ for all $\alpha$.
Note: assuming the existence of $\lim_{\delta\to 0}(-\log N_\delta(\alpha)/\log\delta)$ one may even conclude $\underline{T}(q) \geq F^+(\alpha) - q\alpha$.

**ii)** Fix $q \leq 0$. A similar argument as in i) yields $T(q) \geq F^-(\alpha) - q\alpha$ for $\alpha \geq 0$. Moreover, by direct calculation $T(q) \geq T(0) \geq F^-(\alpha) \geq F^-(\alpha) - q\alpha$ for $\alpha \leq 0$. Hence $T(q) \geq L^-(q)$ and $T(q) \geq l(q)$.
Note: assuming the existence of $\lim_{\delta\to 0}(-\log M_\delta(\alpha)/\log\delta)$ one may even conclude $\underline{T}(q) \geq F^-(\alpha) - q\alpha$.

Now, to estimate $T$ from above, fix $\varepsilon > 0$ and split $G_\delta$ into sets of boxes with $\delta^{k\varepsilon} \leq \mu((B)_1) < \delta^{(k-1)\varepsilon}$. For convenience denote these sets by $G_\delta(k)$. Their cardinality is bounded by $N_\delta(k\varepsilon)$ as well as by $M_\delta((k-1)\varepsilon)$. The appropriate bound will have to be carefully chosen, depending on whether $F^+$ or $F^-$ is estimated and whether $k\varepsilon$ is greater than $\alpha^*$ or not.

**iii)** $T \le L^+$ for $q > 0$: Fix $q > 0$ and take $\varepsilon > 0$. Note that by (1.19) $L^+(q) \in \mathbb{R}$. Choose $m \in \mathbb{N}$ large enough to insure $L^+(q) + qm\varepsilon \ge d$. Then

$$S_\delta(q) = \Big(\sum_{k=0}^{m}\sum_{G_\delta(k)} + \sum_{0\neq\mu((B)_1)<\delta^{m\varepsilon}}\Big)\mu((B)_1)^q \le \sum_{k=0}^{m} N_\delta(k\varepsilon)\delta^{q(k-1)\varepsilon} + \#G_\delta\cdot\delta^{qm\varepsilon}.$$

Choose $\delta_0$ such that

$$N_\delta(k\varepsilon) \le \delta^{-(L^+(q)+qk\varepsilon+\varepsilon)} \quad (k = 0\dots m)$$

for all $0 < \delta < \delta_0$. Then

$$S_\delta(q) \le \sum_{k=0}^{m} \delta^{-(L^+(q)+q\varepsilon+\varepsilon)} + c\delta^{-L^+(q)} \le (m+1+c)\delta^{-(L^+(q)+q\varepsilon+\varepsilon)}$$

for $\delta < \delta_0$, and hence $T(q) \le L^+(q) + q\varepsilon + \varepsilon$. Thereby $\varepsilon > 0$ is arbitrary.

**iv)** $T \le l$ for $q > 0$: If $\alpha^* = \infty$ then $F^+ = F^m$ and by iii) $l = L^+ = T$ for $q > 0$. Otherwise, fix $q \ge 0$ and assume without loss of generality $l(q) \neq \infty$. Take $\varepsilon > 0$ and choose $m \in \mathbb{N}$ such that $m\varepsilon < \alpha^* \le (m+1)\varepsilon$. Then, for $\alpha > m\varepsilon$,

$$\begin{aligned} S_\delta(q) &= \Big(\sum_{k=0}^{m}\sum_{G_\delta(k)} + \sum_{\delta^\alpha\le\mu((B)_1)<\delta^{m\varepsilon}} + \sum_{0\neq\mu((B)_1)<\delta^\alpha}\Big)\mu((B)_1)^q \\ &\le \sum_{k=0}^{m} N_\delta(k\varepsilon)\delta^{q(k-1)\varepsilon} + N_\delta(\alpha)\delta^{qm\varepsilon} + M_\delta(\alpha)\delta^{q\alpha}. \qquad (1.21)\end{aligned}$$

First to the case $F^+(\alpha^*+) \le F^-(\alpha^*-)$. Choose $\alpha > \alpha^*$ such that $F^+(\alpha) \le F^+(\alpha^*+) + \varepsilon$. Then

$$\begin{array}{rclcl} F^+(\alpha) & \le & F^m(\alpha^*) + \varepsilon & \le & l(q) + q\alpha^* + \varepsilon, \\ F^-(\alpha) & \le & F^-(\alpha-) = F^m(\alpha) & \le & l(q) + q\alpha, \\ F^+(k\varepsilon) & \le & F^+(k\varepsilon+) = F^m(k\varepsilon) & \le & l(q) + qk\varepsilon. \end{array}$$

If $l$ and $F^m$ were identically $-\infty$ then by (1.21) $S_\delta(q) = 0$ for small enough $\delta$. This is impossible, thus $l(q) \in \mathbb{R}$. Now there is $\delta_0$ such that

$$N_\delta(k\varepsilon) \le \delta^{-(l(q)+qk\varepsilon+\varepsilon)} \; (k = 0\dots m)$$

and

$$N_\delta(\alpha) \le \delta^{-(l(q)+q\alpha^*+2\varepsilon)}, \quad M_\delta(\alpha) \le \delta^{-(l(q)+q\alpha+\varepsilon)}$$

for all $0 < \delta < \delta_0$. Hence

$$S_\delta(q) \le (m+1)\delta^{-(l(q)+q\varepsilon+\varepsilon)} + \delta^{-(l(q)+q\varepsilon+2\varepsilon)} + \delta^{-(l(q)+\varepsilon)}$$

and $T(q) \le l(q)$. Now, to treat the case $F^+(\alpha^*+) > F^-(\alpha^*-)$, choose $\alpha \in ]m\varepsilon, \alpha^*[$ such that $F^-(\alpha) \le F^-(\alpha^*-) + \varepsilon$. This time $F^+(\alpha) \le F^m(\alpha) \le l(q) + q\alpha$ and $F^-(\alpha) \le l(q) + q\alpha^* + \varepsilon$. The rest goes along similar lines.



**v)** $T \le L^-$ for $q < 0$: unless $F^- \ge 0$, which is treated in (1.20), one may assume the existence of a $\beta$ with $F^-(\beta) = -\infty$. Fix $q \le 0$ and take $\varepsilon > 0$. Choose $m \in \mathbb{N}$ satisfying $m\varepsilon > \beta$. Then $M_\delta(m\varepsilon) = 0$ and, similar as in iii),

$$S_\delta(q) \le \sum_{k=0}^{m} M_\delta((k-1)\varepsilon)\delta^{qk\varepsilon} \le (m+1)\delta^{-(L^-(q)-q\varepsilon+\varepsilon)}$$

for small enough $\delta$, thus $T(q) \le L^-(q)$.

**vi)** $T \le l$ for $q < 0$: as above one may assume $F^-(\beta) = -\infty$. Fix $q \le 0$ and take $\varepsilon > 0$. Since $\alpha^* \le \beta < \infty$ there are integers $n$ and $m$ such that $n\varepsilon > \beta$ and $m\varepsilon > \alpha^* \ge (m-1)\varepsilon$. Then $M_\delta(n\varepsilon) = 0$ for small enough $\delta$ and, provided $\alpha < m\varepsilon$,

$$\begin{aligned} S_\delta(q) &= \Big(\sum_{\delta^\alpha\le\mu((B)_1)} + \sum_{\delta^{m\varepsilon}\le\mu((B)_1)<\delta^\alpha} + \sum_{k=m+1}^{n}\sum_{G_\delta(k)}\Big)\mu((B)_1)^q \\ &\le N_\delta(\alpha)\delta^{q\alpha} + M_\delta(\alpha)\delta^{qm\varepsilon} + \sum_{k=m+1}^{n} M_\delta((k-1)\varepsilon)\delta^{qk\varepsilon}. \end{aligned}$$

Choosing $\alpha \in ]\alpha^* - \varepsilon, \alpha^*[$ such that $F^-(\alpha) \le F^-(\alpha^*-) + \varepsilon$ in the case $F^-(\alpha^*-) \le F^+(\alpha^*+)$, resp. $\alpha \in ]\alpha^*, m\varepsilon[$ such that $F^+(\alpha) \le F^+(\alpha^*+) + \varepsilon$ in the case $F^-(\alpha^*-) > F^+(\alpha^*+)$, a similar argumentation as in iv) yields $T(q) \le l(q)$.

**vii)** For $q \neq 0$ the assertions are proved. Provided $\alpha^* < \infty$ iv) holds for $q = 0$ giving with i): $T(0) = l(0) = \sup_{\alpha\in\mathbb{R}} F^m(\alpha) \le \sup_{\alpha\in\mathbb{R}} F^+(\alpha) \le T(0)$. ◇

The notes in step i) and ii) of the proof above imply:

**Lemma 1.16** *The grid-regularity of $T(q)$ is a consequence of the existence of*

$$\lim_{\delta\downarrow 0}\frac{\log N_\delta(\alpha)}{-\log\delta} \quad (\textit{if } q > 0) \qquad \textit{resp.} \qquad \lim_{\delta\downarrow 0}\frac{\log M_\delta(\alpha)}{-\log\delta} \quad (\textit{if } q < 0),$$

*where $\alpha = -T'(q)$ and continuity of $F^+$ resp. $F^-$ at $\alpha$ is assumed.*

Moreover, proposition 1.15 implies that $T$ is convex:

**Lemma 1.17** *$T$ is certainly continuous, nonincreasing and convex on $\mathbb{R}^+$. Moreover, either these properties hold on all of $\mathbb{R}$, or $T(q)$ is infinite for all $q < 0$.*

**Proof** The monotonicity of $S_\delta(q)$ with respect to $q$ carries over to $T$. Continuity and convexity follow from general properties of Legendre transforms. Furthermore, either $F^-(\alpha) \ge 0$ for all $\alpha$ and $T(q) = \infty$ for all negative $q$ by (1.20). Or $F^-(\alpha) = -\infty$ for $\alpha$ large enough; then $T(q)$ is real-valued for all $q$ and $\alpha^* < \infty$ by (1.19). ◇

Example 2.14 provides a multifractal with grid-regular $T(q)$, which is not even semi-continuous at 0. Consequently for this multifractal $T(q) = \infty$ for all $q < 0$.
Now it shall be investigated under which conditions the singularity exponents $T$ determine the semispectra and hence the spectrum itself. Since there are no regularity conditions imposed on the semispectra, some assumptions must be made on the singularity exponents.

**Theorem 1.1 a)** *If $T$ is differentiable at some $q \neq 0$ and if $\alpha = -T'(q)$, then*

$$F(\alpha) = F^m(\alpha) = T(q) - qT'(q). \tag{1.22}$$

*In particular, $F(\alpha)$ shares the properties of $F^{\pm}(\alpha\pm)$ stated in proposition 1.8.*

**b)** *Assume that $T$ is continuously differentiable in an open interval $U$ and let $V = -T'(U)$. Then, (1.22) holds for all $q \in U$ (including 0) and $F$ restricted to $V$ is strictly concave in the interior of $V$.*
*If, in addition, $\overline{U}$ contains 0, $T$ is continuous in $U \cup \{0\}$ and $T'$ is bounded in $U$, then the equality $F = F^m$ and the continuity of $F$ extend to $\overline{V}$.*
*If, in addition to this, $U$ contains an interval of the form $]0, \varepsilon[$, then $\alpha^* < \infty$.*

**c)** *Provided that $T$ is twice-continuously differentiable in an open interval $U$ with nonvanishing second derivate, then $F$ is differentiable with respect to $\alpha$ at $\alpha = -T'(q)$ with derivate $q$.*

For measures with a situation as described in b) see examples 2.14, 3.6 and [MEH].
**Proof** Again we often use $\max(F^+, F^-) \leq T(0)$ without pointing to it.

**i)** Fix $q > 0$. Apply lemma 1.13 f) to $t = F^+$. Note that $A_1 = 0$ and $A_4 = \infty$ due to $F^+(\alpha) \geq 0$ for $\alpha > d$ and $F^+(\alpha) = -\infty$ for $\alpha < 0$. Set $\bar{\alpha} := x(q) = \lim x_n$. The monotonicity of $F^+$ then implies $F^+(\bar{\alpha}+) \geq \lim_{n\to\infty} F^+(x_n) = T(q) + q\bar{\alpha}$. On the other hand, $F^+(\alpha) \leq T(q) + q\alpha$ for all $\alpha$, which leads to

$$F^+(\alpha+) \leq T(q) + q\alpha \qquad (\alpha \in \mathbb{R},\ q > 0). \tag{1.23}$$

Consequently, $F^+(\alpha_1) \leq T(q) + q\alpha_1 < T(q) + q\bar{\alpha} = F^+(\bar{\alpha}+) \leq F^+(\alpha_2)$ for $\alpha_1 < \bar{\alpha} < \alpha_2$. Thus, $F^+$ is quasi increasing at $\bar{\alpha}$. Proposition 1.10 yields

$$F(\bar{\alpha}) = F^+(\bar{\alpha}+) = T(q) + q\bar{\alpha}. \tag{1.24}$$

To show that $F^+(\bar{\alpha}+) \leq F^-(\bar{\alpha}-)$, assume that there is a $\eta > 0$ such that $F^+(\bar{\alpha}+) \geq F^-(\bar{\alpha}-) + 2\eta$. Note that $F^+(\bar{\alpha}+) = F(\bar{\alpha})$ is real. By lemma 1.13 (with $t = F^m$) there is a sequence $x_n \to \bar{\alpha}$ with $F^m(x_n) \to T(q) + q\bar{\alpha} = F^+(\bar{\alpha}+)$ for $n \to \infty$. By monotonicity of $F^-$ there is $n_0 \in \mathbb{N}$ such that $F^-(x_n) \leq F^-(\bar{\alpha}-) + \eta$ for all $n > n_0$, implying $F^m(x_n) \leq F^-(x_n) \leq F^+(\bar{\alpha}+) - \eta = \lim F^m(x_n) - \eta$. This is a contradiction.
If $q < 0$ is such that $T$ is real-valued in a neighbourhood, then $\bar{\alpha} = x(q)$ is real and by similar arguments as above $F^-(\alpha_1) \geq F^-(\bar{\alpha}-) = T(q) + q\bar{\alpha} > T(q) + q\alpha_2 \geq F^-(\alpha_2)$ for $\alpha_1 < \bar{\alpha} < \alpha_2$. Thus

$$F(\bar{\alpha}) = F^-(\bar{\alpha}-) = T(q) + q\bar{\alpha}, \tag{1.25}$$

which proves (1.22).



Now assume that $T$ is continuously differentiable in $U$. Set $U_1 = U \cap ]0, \infty[$, $U_2 = U \cap ]-\infty, 0[$ and $V_k = -T'(U_k)$ $(k = 1, 2)$. Note that $V_k$ is an interval, not necessarily open.

**ii)** First, take $q_0$ from $U_1$ and set $\bar{\alpha} = x(q_0) = -T'(q_0)$. By (1.23) and (1.24)

$$F(\bar{\alpha}) = F^+(\bar{\alpha}+) = \min_{q\in\mathbb{R}^+}(T(q) + q\bar{\alpha}) = -\sup_{q\in\mathbb{R}^+}(-T(q) - q\bar{\alpha}). \tag{1.26}$$

By lemma 1.13 b) and h) (with $t(q) = -T(q)$ $(q > 0)$, $t(q) = -\infty$ $(q \leq 0)$) $F$ restricted to $V_1$ is continuous. This is trivial if $V_1$ is a singleton. Suppose $F$ were linear in a neighbourhood of $\bar{\alpha} = -T'(q_0)$, contained in $V_1$. Then, again by lemma 1.13 f) and g), $t$ wouldn't be differentiable at the unique touching point $x(\bar{\alpha}) = F'(\bar{\alpha})$. By (1.24) this point is $q_0$, giving a contradiction. Hence $F$ must be strictly concave and strictly increasing in $\mathrm{int}(V_1)$.

**iii)** More can be said. Since $F^+$ is strictly increasing in $\mathrm{int}(V_1)$, it is still quasi increasing at the boundary of $V_1$ and hence $F(\alpha)$ equals $F^+(\alpha+)$ in all of $\overline{V_1}$. In case that $\mathrm{int}(V_1)$ is empty, this is trivial by (1.24). In particular, $F$ is rightcontinuous at the left boundarypoint of $V_1$.

**iv)** The same argumentation applies to $F^-$, showing that $F$ is continuous and equal to $F^-(\alpha-) = F^m(\alpha)$ in $V_2$ and that it is strictly concave and strictly decreasing in $\mathrm{int}(V_2)$. Moreover, it still equals $F^-(\alpha-)$ at the boundary of $V_2$ and is leftcontinuous at the right boundary of $V_2$.

**v)** The case $q = 0$ needs a special treatment, since the Legendre connection is in general not established for $q = 0$, and since the loss of the strict monotonicity inhibits the argumentation of i). However, it is enough to know that $q = 0$ corresponds to the maximum of $F$. The only assumptions are: $0 \in \overline{U}$, continuity of $T$ in $U \cup \{0\}$ and boundedness of its derivate for $q \to 0$ in $U$. This is certainly satisfied when $U$ contains 0.
Assume first $U_1 \neq \emptyset$. By convexity $\bar{\alpha} = -T'(q)$ increases, say to $\alpha_0$, when $q \downarrow 0$. Of course $\alpha_0 \in \overline{V_1}$. If $0 \in U$, then $\alpha_0 = -T'(0)$. From (1.24)

$$F^+(\alpha_0+) \geq \lim_{q\downarrow 0} F^+(\bar{\alpha}+) = T(0),$$

thus actually equality. By iii) $T(0) = F^+(\alpha_0+) = F(\alpha_0)$, which in fact proves $\alpha^* \leq \alpha_0 < \infty$ and establishes (1.22) also for $q = 0$ (provided $U_1 \neq \emptyset$). Moreover, if $V_1 \neq \{\alpha_0\}$ then it was just proven that $F^+(\alpha+)$ is left-continuous at $\alpha_0$.

Thus, $F^+(\alpha_0+) \le F^-(\alpha_0-)$, since $F^+(\alpha+) \le F^-(\alpha-)$ in $V_1$. Consequently,

$$F^+(\alpha_0+) = F^-(\alpha_0-) = T(0). \tag{1.27}$$

Similar $F(\alpha_0) = F^-(\alpha_0-) = T(0)$ for $\alpha_0 = \lim_{q\uparrow 0}(-T'(q))$, provided that $U_2 \ne \emptyset$. Thus, (1.22) is established also for $q = 0$, since $U_1$ and $U_2$ are not both empty.
Moreover, by the strict monotonicity of $F$ in $V_1$ and $V_2$, the spectrum is also in $\alpha_0$ strictly concave and attains there its maximal value. Finally, to prove $F(\alpha_0) = F^m(\alpha_0)$ proceed as follows. If $V = \{\alpha_0\}$ then $\alpha_0 = -T'(q)$ for all $q \in U$ and the claim follows from ii) or iv). Otherwise, $V_1$ and $V_2$ cannot both equal $\{\alpha_0\}$. The assumption $V_2 \ne \{\alpha_0\}$ can be used to prove (1.27) similar as it is carried out with $V_1$. Hence (1.27) is valid, implying what we claimed.

**vi)** Step ii), iv) and v) give the continuity of $F$ and the equality $F = F^m$ at the boundary of $V$ under the assumptions stated in b). An application of the inverse function theorem to $-T'$ completes the proof. ◇

Due to theorem 1.1 the spectrum is pretty well determined by $T$, unless the latter is piecewise linear. In this case only the values at some 'wedge'-points can be obtained. However, in the degenerate case this turns out to be enough:

**Proposition 1.18** *If $T$ is everywhere linear, i.e. $T(q) \equiv T(0) - \alpha_0 q$, then*

$$F(\alpha) = \begin{cases} T(0) & \text{if } \alpha = \alpha_0 \\ -\infty & \text{otherwise.} \end{cases}$$

This comes to its extreme with $T(q) \equiv 0$ for a Dirac measure $\mu$.
**Proof** By (1.23) $F^+(\alpha) \le T(q) + q\alpha = T(0) + q(\alpha - \alpha_0)$ for any $\alpha$ and any positive $q$. This allows to conclude with proposition 1.10 $F(\alpha) = F^+(\alpha) = -\infty$ for $\alpha < \alpha_0$ and $F(\alpha) = F^-(\alpha) = -\infty$ for $\alpha > \alpha_0$. Theorem 1.1 gives $F(\alpha_0) = T(0)$. ◇
Besides concavity there is a further property of $F$ one will typically meet and which arises from the Legendre connection of $T$ and $F$.

**Proposition 1.19 a)** *$D_q$ is positive and nonincreasing for $q > 0$, moreover, it is—except maybe at 1—continuous. Provided $\alpha^* < \infty$, these properties hold in the whole interval in which $D_q$ is real-valued, in particular at 0.*

**b)** *The existence of $\lim_{q\to 1} D_q$ is equivalent with the differentiability of $T$ at 1 and implies the following three facts: $F$ touches the inner bisector of the axes at $D_1 = F(D_1)$,*

$$\lim_{q\to 1} D_q = D_1 = d_1 = -T'(1)$$

*and $D_1$ does not depend on the choice $\kappa = 1$ in the definition 1.11.*



Since the range of $D_q$ is of interest, we let $D_\infty$ denote the infimum and $D_{-\infty}$ the supremum of $D_q$ over all real $q$. In the case $\alpha^* < \infty$ one finds

$$D_\infty := \inf_{q\in\mathbb{R}} D_q = \lim_{q\to\infty} D_q \qquad \text{and} \qquad D_{-\infty} := \sup_{q\in\mathbb{R}} D_q = \lim_{q\to-\infty} D_q,$$

where the value $D_{-\infty} = \infty$ is allowed.
**Proof** From $T(1) = \tau(1) = 0$

$$D_q = -\frac{T(q) - T(1)}{q - 1} \qquad \text{for } q \ne 1.$$

By lemma 1.17 $D_q$ is real-valued for $q \ge 0$, $q \ne 1$. Since it equals the negative slope of the line intersecting the graph of $T$ at 1 and at $q$, it cannot increase and is positive. Now let $0 < q_1 < 1 < q_2$. Fix $\delta > 0$ and $\kappa \ge 0$ and set

$$h(q) := \log\Big(\sum_{B\in G_\delta} \mu((B)_\kappa)^q\Big) = \log(S_\delta(q,\kappa)),$$

supplementing (1.1) with $(B)_0 := B$. Since $G_\delta$ is finite, $h$ is smooth and—by elementary calculus—convex. The mean value theorem of calculus gives

$$\frac{h(q_1) - h(1)}{q_1 - 1} \le h'(1) \le \frac{h(q_2) - h(1)}{q_2 - 1}.$$

Dividing by $\log\delta$ and letting $\delta \to 0$ gives

$$D_{q_1} \ge \limsup_{\delta\downarrow 0} \frac{1}{\log(\delta)} \frac{\sum_{B\in G_\delta} \mu((B)_\kappa)\log\mu((B)_\kappa)}{\sum_{B\in G_\delta} \mu((B)_\kappa)} \ge D_{q_2}. \tag{1.28}$$

Thereby proposition 1.5 and $\underline{T}(1) = \underline{\tau}(1) = 0$ were used. So, monotonicity of $D_q$ holds throughout $q = 1$ and the proof of a) is complete. Trivially $\lim_{q\to 1} D_q = -T'(1)$. Let us assume that this limes exists. It must by (1.28) take the value $D_1$, which does not depend on $\kappa > 0$. With $\kappa = 0$ (1.28) gives $D_1 = d_1$. Furthermore,

$$F(D_1) = T(1) - T'(1) = D_1$$

by theorem 1.1. Observing $N_\delta(\alpha - \varepsilon) \ge 0$ and applying (1.23) with $q = 1$ shows $F(\alpha) \le F^+(\alpha+) \le \alpha$ for all $\alpha$ and the proof is complete. ◇
Comparing the definition of $D_q$ with the one of $D_1$ one might suggest that the latter provides a different and specific information about the multifractal $\mu$. Indeed, $D_1$ is considered to be the most interesting among all $D_q$ [Gr2, GP2]. Apart from proposition 1.19 there are further facts which support the peculiarity of $D_1$. They are listed without intention to be rigorous:

- $D_1$ is the only $D_q$ which remains invariant under a greater family of coordinate transformations [OWY, Koh].

- The measure $\mu$ is concentrated on a set of dimension $D_1$ [Falc4, GH1]. In this context it should be referred to subsection 2.4.4, in particular to the examples 2.10, 2.11 and 2.13 therein.

- Closely related to the latter is the fact, that $D_1$ often equals the Hausdorff dimension of $\mu$. In addition, the Hausdorff dimension of certain self-similar and of some self-affine sets can be obtained as the maximal information dimension $D_1$ of the canonical invariant measures. Compare example 2.6 on page 53 and (3.28) on page 104.

- Amazing too is the fact that for the family of self-affine multifractals in chapter three the differentiability of $T$ can be proved a priori only at $q = 1$.

Finally, some remarks on the grid-regularity of $T(q)$ and $F(\alpha)$ are added. First, remember proposition 1.10 and lemma 1.16. To conclude in the opposite direction one can use the theorem below. It gives the grid-regularity of $F(\alpha)$ and even generalizes (1.26). But it has the disadvantage of not allowing piecewise differentiable functions $T$, which may appear with self-affine multifractals.

**Theorem 1.2** *If $T$ is grid-regular and differentiable on all of $\mathbb{R}$, then*

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\frac{\log(N_\delta(\alpha+\varepsilon)-N_\delta(\alpha-\varepsilon))}{-\log\delta} = F(\alpha) = \inf_{q\in\mathbb{R}}(T(q)+q\alpha) \qquad \forall\alpha\in\mathbb{R}.$$

*In particular $F(\alpha)$ is grid-regular for all $\alpha$. Moreover, $F$ is real-valued exactly in $[D_\infty, D_{-\infty}]$ and continuous there. Thereby, the formula*

$$D_{\pm\infty} = \lim_{q\to\pm\infty} -T'(q)$$

*is valid.*

**Proof** Write $l(\alpha) = -\inf_{q\in\mathbb{R}}(T(q)+q\alpha) = \sup_{q\in\mathbb{R}}(-T(q)-q\alpha)$ for short.

**i)** $T$ is convex for $q > 0$ and for $q < 0$. Since it is differentiable at 0, it has to be convex there as well.

**ii)** Now take any sequence $(\delta_n)_{n\in\mathbb{N}}$ of positive numbers, tending to zero, and fix it. In order to apply Ellis' theorem II.2 [Ell, page 3] supply $G_{\delta_n}$ with the uniform distribution denoted by $P_n$. Define the random variable $Y_n(B) := \log(\mu((B)_1)$ on $G_{\delta_n}$ and calculate its moment generating function:

$$E_n[e^{qY_n}] = \frac{1}{\#G_{\delta_n}}\sum_{B\in G_{\delta_n}}\mu((B)_1)^q = \frac{1}{S_{\delta_n}(0)}S_{\delta_n}(q).$$

Next define $a_n := -\log\delta_n$ and

$$c_n(q) := \frac{1}{a_n}\log E_n[e^{qY_n}] = \frac{\log S_{\delta_n}(q)}{-\log\delta_n} - \frac{\log S_{\delta_n}(0)}{-\log\delta_n}.$$



The hypothesis of the theorem to be proven and i) imply, that

$$c(q) := \lim_{n\to\infty} c_n(q) = T(q) - T(0)$$

exists and is a convex function, differentiable on all of $\mathbb{R}$. Thus, the hypotheses of Ellis' theorem are satisfied.

**iii)** Define now

$$I(z) := \sup_{q\in\mathbb{R}}(qz - c(q)) = -\inf_{q\in\mathbb{R}}(T(q) - qz - T(0)) = T(0) + l(-z)$$

and set $H := [-\alpha,\infty[\,, G := ]-\alpha,\infty[$ for a fixed, but arbitrary $\alpha$. Then

$$P_n[\frac{1}{a_n}Y_n\in G] \le P_n[\frac{1}{a_n}Y_n\in H] = P_n[Y_n \ge \log{\delta_n}^\alpha] = \frac{1}{S_{\delta_n}(0)}N_{\delta_n}(\alpha).$$

Ellis' theorem II.2 now reads as

$$-\inf\{I(z)\ : z\in H\} \ge \limsup_{n\to\infty}\frac{1}{a_n}\log P_n[\tfrac{Y_n}{a_n}\in H] = \limsup_{n\to\infty}\frac{\log N_{\delta_n}(\alpha)}{-\log\delta_n} - T(0),$$

$$-\inf\{I(z)\ : z\in G\} \le \liminf_{n\to\infty}\frac{1}{a_n}\log P_n[\tfrac{Y_n}{a_n}\in G] \le \liminf_{n\to\infty}\frac{\log N_{\delta_n}(\alpha)}{-\log\delta_n} - T(0).$$

**iv)** Applying lemma 1.13 to the function $t = -T$ gives $A_1,\ldots,A_4$ with the obvious properties. Thereby, $A_2$ and $A_3$ coincide because $t$ is differentiable at 0. Furthermore, since $T$ is differentiable and convex, it must be continuously differentiable. The expressions for $D_\infty$ and $D_{-\infty}$ follow now from their definitions, and the explicit formula for the Legendre transform $l(t'(q)) = t(q) - qt'(q)$ gives $A_1 = D_\infty$, $A_4 = D_{-\infty}$. Even $F = -l$ is now established in $[D_\infty, D_{-\infty}]$, but not yet the grid-regularity. By continuity of $l$

$$\inf\{I(z)\ : z\in H\} = \inf\{I(z)\ : z\in G\} = \begin{cases}\infty = l(\alpha) & \text{if } \alpha < A_1,\\ T(0)+l(\alpha) & \text{if } A_1 < \alpha \le A_2,\\ T(0)+l(A_2) & \text{if } A_2 < \alpha.\end{cases}$$

The value $\alpha = A_1$ has to be omitted to guarantee the first equality.

**v)** Since the sequence $\delta_n$ was arbitrary, iii) and iv) imply

$$F^+(\alpha) = \lim_{\delta\downarrow 0}\frac{\log N_\delta(\alpha)}{-\log\delta} = \begin{cases}-l(\alpha) & \text{if } \alpha\le A_2,\ \alpha\ne A_1\\ -l(A_2) & \text{if } \alpha > A_2.\end{cases}$$

Moreover, this function is strictly monotonous increasing in $]A_1, A_2[$. Thus, the additional precondition of proposition 1.10 a) is satisfied for any $\alpha\in\,]-\infty, A_2]$, giving the grid-regularity of $F(\alpha)$ and its value

$$F(\alpha) = F^+(\alpha+) = \left\{\begin{array}{ll} = -l(\alpha+) & \text{if } \alpha < A_2\\ = -l(A_2) & \text{if } \alpha = A_2\end{array}\right\} = -l(\alpha).$$

Note that also the special value $\alpha = A_1$ (lemma 1.13 h) and the special case $A_1 = A_2$ are covered.

**vi)** Changing only the definitions of $H$ and $G$ to $H := ]-\infty, -\alpha]$ and $G := ]-\infty, -\alpha[$ results in

$$P_n[\frac{1}{a_n}Y_n \in H] \geq P_n[\frac{1}{a_n}Y_n \in G] = P_n[Y_n < \log {\delta_n}^{\alpha}] = \frac{1}{S_{\delta_n}(0)} M_{\delta_n}(\alpha).$$

Ellis' theorem gives

$$-\inf\{I(z) \ : z \in H\} \geq \limsup_{n\to\infty} \frac{1}{a_n} \log P_n[\frac{Y_n}{a_n} \in H] \geq \limsup_{n\to\infty} \frac{\log M_{\delta_n}(\alpha)}{-\log \delta_n} - T(0)$$

and

$$-\inf\{I(z) \ : z \in G\} \leq \liminf_{n\to\infty} \frac{1}{a_n} \log P_n[\frac{Y_n}{a_n} \in G] = \liminf_{n\to\infty} \frac{\log M_{\delta_n}(\alpha)}{-\log \delta_n} - T(0).$$

Since $A_2 = A_3$, one finds

$$\inf\{I(z) \ : z \in H\} = \inf\{I(z) \ : z \in G\} = \begin{cases} T(0) + l(A_2) & \text{if } \alpha < A_2, \\ T(0) + l(\alpha) & \text{if } A_2 \leq \alpha < A_4, \\ \infty = l(\alpha) & \text{if } A_4 < \alpha. \end{cases}$$

A similar argument as above gives the grid-regularity of $F(\alpha)$ and its value

$$F(\alpha) = F^-(\alpha-) = -l(\alpha)$$

for all $\alpha \geq A_2$. This completes the proof. $\diamond$

This proof revealed some facts about the semispectra:

**Lemma 1.20** *If $T$ is grid-regular and differentiable on all of $\mathbb{R}$, then $F(\alpha) = F^m(\alpha)$ for all $\alpha$ and*

$$F^+(\alpha) = \lim_{\delta\downarrow 0} \frac{\log N_\delta(\alpha)}{-\log \delta} \quad (\alpha \neq A_1) \qquad F^-(\alpha) = \lim_{\delta\downarrow 0} \frac{\log M_\delta(\alpha)}{-\log \delta} \quad (\alpha \neq A_4).$$

# Chapter 2

# Self-Similar Multifractals

This chapter is devoted to the multifractals arising from a generalization of the Cantor set construction. Sometimes this kind of construction is referred to as a 'multiplicative cascade' [EM] or as a 'Moran construction' [CM]. For the resulting measures we will use the short form CMF, which may be read as Cantor Multifractal or Cascade Multifractal. Though all CMFs will share a common basic structure, their diversity is great enough to fit in the various applications [HP, V1, TV, GS, Tél]. In section one we give the definition of CMFs and prove some properties they have in common. In section two we present a short survey of a special case of this construction: the Iterated Function Systems IFS, which are widely used and studied [Bar, Falc4, Falc3, BEH, BEHM, Bed3, Bed4, GH2, GM, Ma1, Ma2]. Again a special kind of IFS are the well-known self-similar measures. The computation of their multifractal spectrum is carried out in section three and applications follow in section four.

## 2.1 Cantor Set and Codespace

First, the usual formalism in connection with Cantor sets and symbolic dynamics is introduced. For a full treatment of the statements made in this section [Hut] is a good reference.
Fix a natural number $r$. To design a so-called *$r$-adic Cantor set* $K$ one generalizes the construction carried out in example 1.1. Take a compact subset $V$ of $\mathbb{R}^d$ and choose $r$ closed subsets $V_1, \dots, V_r$ of $V$, not necessarily disjoint. Now go on like this, replacing $V$ by $V_k$ and denoting the subsets of $V_k$ by $V_{k1} \dots V_{kr}$. So, inductively $r$ closed subsets $V_{\underline{i}*k}$ $(k = 1, \dots, r)$ of $V_{\underline{i}}$ are obtained, where

$$\underline{i} := i_1 \dots i_n \in I_n := \{1, \dots, r\}^n \quad \text{and} \quad \underline{i} * k := i_1 \dots i_n k$$

for short. We will address $\underline{i}$ as a *finite word*, or just *word*, of *length* $|\underline{i}| := n$, and denote by $I = \cup_{n\in\mathbb{N}} I_n$ the set of all words. Moreover, it is convenient to introduce the *empty word* nil, and to define nil $* k := k$ and $V_{\text{nil}} := V$. The sets $V_{\underline{i}}$ may overlap, but

$$\max\{\operatorname{diam}(V_{\underline{i}}) : \underline{i} \in I_n\} \to 0 \qquad (n \to \infty)$$

is required. Then define

$$K_n := \bigcup_{\underline{i} \in I_n} V_{\underline{i}} \qquad \text{and} \qquad K := \bigcap_{n \in \mathbb{N}} K_n. \tag{2.1}$$

The sequence $K_n$ is sometimes addressed as a *cascade*. Since it is a decreasing sequence of compact sets, $K$ is compact too and not empty. It is often a fractal—in the sense of $d_{\text{box}}(K) \notin \mathbb{N}$—and carries a rich geometrical structure.
The decreasing diameters of the sets $V_{\underline{i}}$ enable one to codify the points of $K$. Call

$$I_\infty := \{\underline{i}_\infty = i_1 i_2 \ldots : i_k \in \{1, \ldots, r\}, k \in \mathbb{N}\} = \{1, \ldots, r\}^{\mathbb{N}}$$

the *codespace* and set

$$(\underline{i}_\infty | n) := i_1 \ldots i_n \in I_n.$$

For fixed $\underline{i}_\infty$ the sets $V_{(\underline{i}_\infty|n)}$ $(n \in \mathbb{N})$ build a decreasing sequence of compact sets. Thus their intersection will not be empty. Since their diameter decreases to zero, this intersection is a singleton, say $\{x_{\underline{i}_\infty}\}$, and the *coordinate map*

$$\pi : I_\infty \to K \quad \underline{i}_\infty \mapsto x_{\underline{i}_\infty}$$

is continuous and surjectiv ($I_\infty$ carries the product topology of the discrete spaces $\{1, \ldots, r\}$). Moreover, if the sets $V_{\underline{i}}$ $(\underline{i} \in I_n)$ are mutually disjoint for $n$ large enough, then $\pi$ provides a homeomorphism of the topological spaces $I_\infty$ and $K$.
Now a measure $\mu$ is introduced which is supported by $K$ and which carries information about the construction of $K$. Let $(p_1, \ldots, p_r)$ be a *probability vector*, i.e. $p_i > 0$ and $p_1 + \ldots + p_r = 1$, and let $P$ be the product measure on $I_\infty$ induced by the measure $\{j\} \mapsto p_j$ on the factors $\{1 \ldots r\}$, i.e.

$$P[\{\underline{i}_\infty \in I_\infty : i_{k_m} = j_m, \ m = 1 \ldots n\}] = p_{j_1} \cdot \ldots \cdot p_{j_n} =: p_{\underline{j}}$$

for all $n \in \mathbb{N}$, all words $\underline{j}$ of length $n$ and all integers $k_1 < \ldots < k_n$. This measure $P$ exists due to Kolmogorov's consistency theorem [Pth, p 144]. It is Borelsch and $P[I_\infty] = 1$. Thus the measure $\mu := \pi_* P$ (i.e. $\mu(A) = P[\pi^{-1}(A)]$ ) has total mass $\mu(\mathbb{R}^d) = 1$ and at least the Borel sets of $\mathbb{R}^d$ are measurable. Moreover,

$$\mu(V_{\underline{i}}) = P[\{\underline{j}_\infty \in I_\infty : \pi(\underline{j}_\infty) \in V_{\underline{i}}\}] \geq P[\{\underline{j}_\infty \in I_\infty : (\underline{j}_\infty | n) = \underline{i}\}] = p_{\underline{i}} \tag{2.2}$$

with equality holding certainly if $V_{\underline{i}}$ does not intersect any $V_{\underline{k}}$ with $|\underline{k}| = |\underline{i}|$. For later use note, that if equality does hold in (2.2) for all finite words $\underline{i}$ of sufficiently large length, then any singleton is a null set.
From (2.2) it is easy to see, that $\mu$ is supported by $K$. For if $x$ would lie in $K$ but not in $\operatorname{supp}(\mu)$, there would be a neighbourhood $W$ of $x$ which would not meet



$\operatorname{supp}(\mu)$. But since there is also a set $V_{\underline{i}}$ contained in $W$ and $\mu(V_{\underline{i}}) \geq p_{\underline{i}} \neq 0$ this is not possible. The reverse, i.e. $\operatorname{supp}(\mu) \subset K$, is immediate.
Thus $\mu$ is a multifractal, uniquely determined by the coordinate map and the probability vector. To express this we introduce the following notation:

$$\mu = \langle \pi; p_1, \ldots, p_r \rangle. \tag{2.3}$$

**Definition 2.1** *A multilfractal $\mu = \langle \pi\,; p_1, \ldots, p_r \rangle$ constructed as above will be called* Cantor Multifractal*, for short* CMF*. Its support is $K = \pi(I_\infty)$.*

It is also common to say that $\mu$ is constructed by a *multiplicative cascade*, referring to the product structure of $P$ as well as to the construction of $K$.
At this point we would like to stop for a moment and explain what is meant by a multifractal formalism *'tailored to multiplicative cascades'*. This simply means that in the definitions 1.4 and 1.7 of $S_\delta(q)$, $N_\delta(\alpha)$ and $M_\delta(\alpha)$ coverings by $\delta$-boxes are replaced by coverings consisting of the *cylindrical sets* $V_{\underline{i}}$ with $|\underline{i}| = n$.
By definition, this multifractal formalism does not distinguish between $\mu$ on $\mathbb{R}^d$ and $P$ on $I_\infty$. On the one hand, this allows the use of symbolic dynamics, which is most effective [BR, CM]. On the other hand, it lacks geometrical relevance since $\pi$ is not involved.
This relevance is provided by the approach proposed in this thesis. Compare the proofs of our main theorems 2.3 and 3.3 and also example 2.10.
All CMFs have some regularity properties in common. A first one is the following:

**Lemma 2.2** *Given any measure $\mu$, substituting the condition '$\mu(B) \neq 0$' in the definition of $S_\delta$, $N_\delta$ and $M_\delta$ (see definitions 1.4 and 1.7) by the condition '$B \cap K \neq \emptyset$' will not affect the values of $T(q)$, $\underline{T}(q)$, $F^+(\alpha+)$ and $F^-(\alpha-)$.*

This might be important in numerical simulations. In particular for $q = 0$:

**Corollary 2.1** *The box dimension of the support of any measure $\mu$ equals*

$$\overline{d_{\text{box}}}(K) = T(0) = D_0$$

*and $d_{\text{box}}(K)$ exists exactly when $T(0)$ is grid-regular.*

**Proof** Let $B$ denote a $\delta$-box. Then, since $B \cap K \neq \emptyset$ is a stronger requirement than $\mu(B) \neq 0$,

$$\begin{aligned} S_\delta(q) &\leq \sum_{B \cap K \neq \emptyset} \mu((B)_1)^q \\ N_\delta(\alpha) &\leq \#\{B : B \cap K \neq \emptyset, \ \mu((B)_1) \geq \delta^\alpha\} \\ M_\delta(\alpha) &\leq \#\{B : B \cap K \neq \emptyset, \ \mu((B)_1) < \delta^\alpha\}. \end{aligned}$$

On the other hand, if $B \cap K \neq \emptyset$ holds, then by the very definition of the support of a measure

$$\mu((B)_{1/2}) \neq 0.$$

Thus, there exists $C_B \in G_{\delta/2}$ and $D_B \in G_{2\delta}$, i.e. with nonvanishing measure, both meeting $(B)_{1/2}$. Since

$$(C_B)_1 \subset (B)_1 \quad , \quad (D_B)_1 \supset (B)_1,$$

and at most $2^d$, resp. $5^d$ $\delta$-boxes $B$ can share the same fixed $C$ from $G_{\delta/2}$ as $C_B$ resp. the same fixed $D$ from $G_{2\delta}$ as $D_B$, the argumentation of the proofs of propositions 1.5 and 1.8 applies. $\diamond$

Often there exist $r$ numbers $\lambda_i \in ]0,1[$ $(i = 1,\ldots,r)$ such that:

$$\frac{\operatorname{diam} V_{\underline{i}}}{\operatorname{diam} V} \le \lambda_{\underline{i}} := \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_n}$$

for all words $\underline{i} = i_1 \ldots i_n \in I$. If this is the case, the Cantor set and any resulting CMF will be called *contractive with* $\lambda_1,\ldots,\lambda_r$. A sufficient, but not necessary condition is

$$\sup_{\underline{i}} \frac{\operatorname{diam} V_{\underline{i}*j}}{\operatorname{diam} V_{\underline{i}}} =: \lambda_j < 1 \qquad (j = 1\ldots r),$$

where the supremum is taken over all finite words $\underline{i}$.

**Lemma 2.3** *If the CMF* $\langle \pi\,;\, p_1,\ldots,p_r\rangle$ *is contractive with* $\lambda_1,\ldots,\lambda_r$ *then*

$$F^-(\alpha) = -\infty \qquad \forall \alpha > \beta := \max_{i=1\ldots r} \frac{\log p_i}{\log \lambda_i}.$$

For the proof as well as for later use certain sets $J_\delta$ are needed. Assume that numbers $\lambda_i \in ]0,1[$ $(i = 1,\ldots,r)$ are given. Roughly speaking $J_\delta$ is the set of all words $\underline{i}$ with $\lambda_{\underline{i}} \simeq \delta$. It will be constructed recursively. Fix $\delta > 0$ and start with $J(1) := I_1$. Suppose $J(m)$ has been constructed and consider an arbitrary word $\underline{i} = i_1 \ldots i_n$ from $J(m)$. If $\lambda_{\underline{i}} \le \delta$ then let $\underline{i}$ be a member of $J(m+1)$, otherwise let $\underline{i}*k$, $(k = 1,\ldots,r)$ belong to $J(m+1)$. Do so for all words of $J(m)$ and add no further words to $J(m+1)$. This defines $J(m+1)$ uniquely. Since all $\lambda_i < 1$, $J(m) = J(m_0)$ for large enough $m_0$ and all $m \ge m_0$. This set of words is the desired $J_\delta$. Due to its construction it has the following property: provided $\delta < \min(\lambda_1,\ldots,\lambda_r)$,

$$J_\delta := J(m_0) = \{\underline{i} = i_1 \ldots i_n \in I \,:\, \lambda_{\underline{i}} \le \delta < \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_{n-1}}\}. \tag{2.4}$$

Moreover, $J_\delta$ is *tight* [Hut], i.e. if $\underline{i}$ is contained in $J_\delta$ then no word of the form $\underline{i} * \underline{k}$ ($\underline{k} \ne$ nil) will belong to $J_\delta$—or to say it positively:

$$\underline{i} \ne \underline{j} \in J_\delta \quad \Rightarrow \quad \exists k \le \min(|\underline{i}|,|\underline{j}|) \,:\, i_k \ne j_k. \tag{2.5}$$

The latter will be abbreviated by $\underline{i}\#\underline{j}$. Also of importance is that $J_\delta$ is *secure*, i.e.

$$K \subset \bigcup_{\underline{i} \in J_\delta} V_{\underline{i}}. \tag{2.6}$$



Equivalently, 'secure' means: for any $\underline{j}_\infty \in I_\infty$ there is $n \in \mathbb{N}$ and $\underline{i} \in J_\delta$ with $\underline{i} = (\underline{j}_\infty|n)$. Note that $n$ and $\underline{i}$ are unique because $J_\delta$ is tight.

**Proof of Lemma 2.3** For simplicity assume $\operatorname{diam}(V) = 1$ (Prop. 1.5). Let $\alpha > \beta$ and set $\underline{\lambda} = \min\{\lambda_1,\ldots,\lambda_r\}$. Fix $\varepsilon > 0$ such that $\alpha - \varepsilon > \beta$. Take $\delta \in ]0, \underline{\lambda}^{(\alpha-\varepsilon)/\varepsilon}[$ and $B \in G_\delta$. By (2.6) there is $\underline{i} \in J_\delta$ s.t. $V_{\underline{i}}$ and $B$ meet. Since $\operatorname{diam} V_{\underline{i}} \le \lambda_{\underline{i}} \le \delta$, $V_{\underline{i}}$ is contained in $(B)_1$. So, observing (2.2) and (2.4),

$$\mu((B)_1) \ge p_{\underline{i}} \ge \lambda_{\underline{i}}^{\alpha-\varepsilon} \ge (\underline{\lambda}\delta)^{\alpha-\varepsilon} \ge \delta^\alpha$$

and $M_\delta(\alpha) = 0$. This holds for small enough $\delta$, leading to $F^-(\alpha) = -\infty$. The proof is complete, but let us draw some further consequences. First, $\alpha^* \le \beta < \infty$, which calls for an application of lemma 1.17. Moreover, since $\mu((B)_1) \ge \delta^\alpha$ for all $B$,

$$S_\delta(q) \le \#G_\delta \cdot \delta^{q\alpha} = S_\delta(0) \cdot \delta^{q\alpha}$$

for $q < 0$ and

$$T(q) \le \limsup_{\delta \downarrow 0} \frac{\log S_\delta(0) + q\alpha \log \delta}{-\log \delta} = T(0) - q\alpha.$$

Since $T(0) \le d$, $T(q)$ must be real. By proposition 1.19 $D_q$ increases as $q$ decreases to $-\infty$, and because $\alpha > \beta$ is arbitrary, the generalized dimensions are bounded by

$$D_{-\infty} = \lim_{q\to-\infty} D_q = \lim_{q\to-\infty} \frac{T(q)}{1-q} \le \lim_{q\to-\infty} \frac{T(0) - q\beta}{1-q} = \beta.$$

This is summarized in the following corollary. $\diamond$

**Corollary 2.2** *If the CMF* $\mu = \langle \pi\,;\, p_1,\ldots,p_r\rangle$ *is contractive with* $\lambda_1,\ldots,\lambda_r$*, then* $T$ *is the Legendre transform of* $F^m$*, real-valued, convex and continuous on all of* $\mathbb{R}$*. Moreover,*

$$0 \le D_\infty \le D_{-\infty} \le \beta$$

*with* $\beta$ *from lemma 2.3.*

Considering corollary 2.7 these bounds cannot be improved within this generality. To relate this result about our fractal formalism with the one presented in section 1.1, it should be referred back to the example 1.1: the former singularity exponents $\tau$ are infinite for negative $q$, even for contractive CMFs. So these infinite exponents do not reflect a geometrical property of the measure $\mu$, such as the occurrence of arbitrarily large local Hölder exponents, but a defect in the method of measurement. Below a CMF will be constructed for which also the former spectrum $f(\alpha)$ does not provide the intended kind of information.

**Example 2.1 (Old Spectrum $f(\alpha)$ versus New Spectrum $F(\alpha)$)** Given $r \geq 2$, $(p_1, \ldots, p_r)$ and $\lambda \in ]0, (r+2)^{-1}[$, a CMF on [0,1] will be constructed with

$$f^-(\alpha) \equiv d_{\text{box}}(K) = -\frac{\log r}{\log \lambda}.$$

This $f$ makes one believe that there should be local Hölder exponents of arbitrary large size. However, the new formalism, which is considered to represent local behaviour in a more accurate manner, yields the grid-regular

$$T(q) = \frac{\log\left(\sum_{i=1}^r p_i^q\right)}{-\log \lambda} \qquad \forall q \in \mathbb{R}$$

for this CMF (theorem 2.3). This agrees with the geometrical intuition involved in the construction of the multifractal to be carried out below. Theorem 2.6 then establishes the grid-regular $F$ as the Legendre transform of $T$ (see also Fig. 2.1).

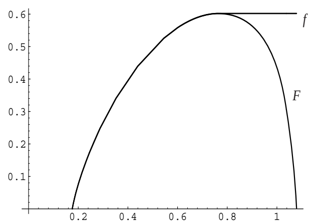


Figure 2.1: *New and old spectrum for a CMF constructed in example 2.1 with* $r = 4$, $p_1 = 2/3$, $p_2 = 1/8$, $p_3 = 1/8$, $p_4 = 1/12$, $\lambda = 1/10$.

Start with $V_{\text{nil}} = [0,1]$. In each step of the construction $r$ disjoint, closed subintervals $V_{\underline{i}*m}$ of $V_{\underline{i}}$ will be chosen, each of length $\lambda^{|\underline{i}|+1}$ and carefully positioned. Assume that $V_{\underline{i}}$ has been constructed. Write $n$ for $|\underline{i}|$. Set $V_{\underline{i}*1}$ to be the closed interval of length $\lambda^{n+1}$ which has its left boundary point in common with $V_{\underline{i}}$ (see Fig. 2.2). Since this choice remains the same through all stages of the construction, the interval with left boundary point in common with $V_{\underline{i}}$ and length $\lambda^{n+k}$ is $V_{\underline{j}}$, where $\underline{j}$ equals $\underline{i}$ followed by $k$ letters 1. Thus, the construction finished, $V_{\underline{j}}$ will by disjointness carry exactly the measure $p_{\underline{i}} \cdot {p_1}^k$. This is very useful to know: since the other subintervals $V_{\underline{i}*2}$, .., $V_{\underline{i}*r}$ are arranged without a common rule, it is impossible to predict where else exactly in $V_{\underline{i}}$ one will find points of the support $K$ of $\mu$. To define the remaining $r-1$ subintervals choose first $r-2$ disjoint closed subintervals $V_{\underline{i}*3}$, ..., $V_{\underline{i}*r}$ of $V_{\underline{i}}$. Since $\lambda(r-1) < 1-3\lambda$, it is possible to arrange them in a way to leave a subinterval of $V_{\underline{i}}$ of length greater than $3 \cdot \lambda^{n+1}$ at one's disposal, which intersects none of the so far constructed $V_{\underline{i}*m}$ $(m \neq 2)$ (see Fig. 2.2). Thus there is $l_{\underline{i}} \in \mathbb{N}$ such that $[(l_{\underline{i}}-1)\lambda^{n+1}, (l_{\underline{i}}+1)\lambda^{n+1}[$ lies in $V_{\underline{i}}$ and meets none of $V_{\underline{i}*m}$ $(m \neq 2)$. Define

$$V_{\underline{i}*2} := [l_{\underline{i}}\lambda^{n+1} - \lambda^{n+1+k_n}, (l_{\underline{i}}+1)\lambda^{n+1} - \lambda^{n+1+k_n}]$$



where $k_n$ has to be chosen large enough to satisfy the inequality

$$p_1^{k_n} < \lambda^{n(n+1)}.$$

Then the $V_{\underline{i}*m}$ are disjoint and the measure of $B_{\underline{i}} := [(l_{\underline{i}}-1)\lambda^{n+1}, l_{\underline{i}}\lambda^{n+1}[$ amounts exactly $p_{\underline{i}} \cdot p_1^{k_n}$. The construction is complete. Moreover, corollary 2.4 applies with $\lambda_i = \lambda$ yielding the announced $T(q)$.
Finally let us calculate $f^-(\alpha)$. Let $\delta_n = \lambda^{(n+1)}$. Fix $\alpha > 0$ and take $n \geq \alpha$, $n \in \mathbb{N}$. For each $\underline{i}$ with $|\underline{i}| = n$ the $\delta_n$-box $B_{\underline{i}}$ carries by construction the measure $p_{\underline{i}} \cdot p_1^{k_n} < {\delta_n}^n$. Since $B_{\underline{i}}$ lies in $V_{\underline{i}}$ these boxes are mutually disjoint. So, their number satisfies

$$m_{\delta_n}(\alpha) \geq \# I_n = r^n$$

and with corollary 2.1

$$f^-(\alpha) \geq \limsup_{n\to\infty} \frac{\log m_{\delta_n}(\alpha)}{-\log \delta_n} \geq -\frac{\log r}{\log \lambda} = T(0) = d_{\text{box}}(K).$$

The reverse inequality follows immediately from lemma 1.9. ○

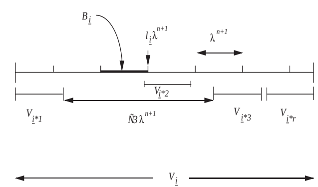


Figure 2.2: *Construction of example 2.1.*

## 2.2 Iterated Function Systems

One way to set up the construction of an r-adic Cantor set is to use a set of strict contractions $(w_1, \ldots, w_r)$ of $\mathbb{R}^d$. Each $w_i$ possesses a unique fixpoint. Thus a sufficiently large open ball $O$ satisfies $w_i(O) \subset O$ for $i = 1, \ldots, r$. Letting

$$V_{\text{nil}} := \overline{O} \qquad \text{and} \quad V_{\underline{i}} := w_{\underline{i}}(\overline{O}) := w_{i_1} \circ \ldots \circ w_{i_n}(\overline{O}) \tag{2.7}$$

yields $V_{\underline{i}*k} = w_{\underline{i}*k}(\overline{O}) = w_{\underline{i}}(V_k) \subset w_{\underline{i}}(\overline{O}) = V_{\underline{i}}$. Of course the diameter of $V_{\underline{i}}$ tend to zero with $|\underline{i}| \to \infty$, moreover, the resulting r-adic Cantor set is contractive with

$$\lambda_i = \text{Lip}(w_i) := \sup_{x \neq y} \frac{|w_i(x) - w_i(y)|}{|x-y|} < 1 \quad (i = 1, \ldots, r).$$

This kind of construction is called an *Iterated Function System*, for short *IFS*.
It is essential to regard the IFS from the point of view of the contraction theorem [Hut, Bar, BEH]. The family $\mathcal{K}$ of all nonempty compact subsets of $\mathbb{R}^d$ becomes a complete [Falc1, p 37] metric space when supplied with the *Hausdorff metric* (i.e. $\rho(A,B) = \sup\{\text{dist}(a,B), \text{dist}(b,A) : a \in A, b \in B\}$, where $\text{dist}(x,A) := \inf\{|x-a| : a \in A\}$). The set mapping

$$W : A \mapsto \bigcup_{i=1}^{r} w_i(A)$$

is contractive since $\rho(W(A), W(B)) \leq \max\{\lambda_1, \ldots, \lambda_r\} \cdot \rho(A,B)$ [BEH, p 12]. Hence $W$ possesses a unique fixpoint in $\mathcal{K}$ and for any choice of a compact set $A$ the sequence $W^n(A)$ converges in Hausdorff metric to this fixpoint. If one chooses $A$ to be $\overline{O}$, then $W^n(A) = K_n \supset K$ (see (2.1)) and hence

$$\rho(K, W^n(A)) \leq \max\{\text{diam} V_{\underline{i}} : \underline{i} \in I_n\} \leq (\max\{\lambda_1, \ldots, \lambda_r\})^n \cdot \text{diam}(V_{\text{nil}}) \to 0.$$

Thus the fixpoint of $W$ is just $K$. Moreover, the completeness of $\mathcal{K}$ is actually not needed, since the fixpoint can be explicitly constructed (see also [Falc4, p 114]). However,

$$K = \bigcup_{i=1}^{r} w_i(K) \tag{2.8}$$

and $K$ only depends on $(w_1, \ldots, w_r)$ and not on the choice of $O$. This is expressed by using the notation

$$K = \langle w_1, \ldots, w_r \rangle \qquad \mu = \langle w_1, \ldots, w_r; p_1, \ldots, p_r \rangle. \tag{2.9}$$

The above construction by $K_n = W^n(\overline{O})$ gives an approximation of $K$ 'from above', which can be realized on a computer screen. There are other ways to get a 'picture' of $K$: choose $i \in \{1, \ldots, r\}$ arbitrarily and consider $A = \{a_i\}$, where $a_i$ is the fixpoint of $w_i$. Obviously $a_i$ is contained in $W^n(A)$ for all $n$. Hence it lies in the closure of $K$, which is $K$ itself. By (2.8) the sets $W^n(A)$ form an increasing sequence of subsets of $K$. Since they converge to $K$ in the Hausdorff metric, their union must be dense in $K$. This gives a deterministic algorithm for 'drawing' $K$. For applications Barnsley [Bar] is a good reference.
To get a random algorithm choose an arbitrary probability vector $(p_1, \ldots, p_r)$ and supplement the IFS $K$ to a CMF. Then pick $\underline{i}_\infty = i_1 i_2 \ldots \in I_\infty$ at random according to the distribution $P$, i.e. $P[i_k = l] = p_l$ for all $k$. Finally take any $x \in \mathbb{R}^d$ and define

$$x_n := w_{i_n} \circ \ldots \circ w_{i_1}(x).$$

By the Ergodic theorem (see [BEH, p 6]) the average visiting time of a box $B$ (i.e. $1/n \cdot \#\{x_k \in B : 1 \leq k \leq n\}$) then approximates $\mu(B)$ for every choice of $x$ and $P$-almost every $\underline{i}_\infty$. As a consequence $x_n$ tends to $K$ (i.e. $\text{dist}(x_n, K) \to 0$) and $K$



is contained in the closure of the union of all $x_n$. From this it is easy to derive a random algorithm.
Note that in contrary to $x_n$, the sequence

$$y_n := w_{i_1} \circ \ldots \circ w_{i_n}(y)$$

converges to $\pi(\underline{i}_\infty)$ (since $y_n \in V_{(\underline{i}_\infty|n)}$) for any choice of $y \in \overline{O}$ and any $\underline{i}_\infty$. An illuminating way to see the difference between the forward and the backward orbits is to translate the corresponding iteration into a dynamical system on the codespace. For the choice $x = y = \pi(\underline{j}_\infty) \in K$ the sequence of infinite words

$$(i_n i_{n-1} \ldots i_2 i_1 j_1 j_2 j_3 \ldots)_{n \in \mathbb{N}}$$

is by $\pi$ mapped onto the sequence $(x_n)_{n \in \mathbb{N}}$, while

$$(i_1 i_2 \ldots i_{n-1} i_n j_1 j_2 j_3 \ldots)_{n \in \mathbb{N}}$$

is mapped onto $(y_n)_{n \in \mathbb{N}}$. The first one has almost surely fluctuating initial segments, while the second one obviously converges to $\underline{i}_\infty$ in the product topology.
More can be said about the approximation of $\mu$. The space of all probability measures with compact support can be supplied with a metric which induces exactly the weak topology [Hut, p 732]. The CMF $\mu = \langle w_1, \ldots, w_r; p_1, \ldots, p_r \rangle$ is the unique fixpoint of the contraction map

$$M : \nu \mapsto \sum_{i=1}^{r} p_i \cdot w_{i*}\nu$$

on this space [Hut, theorem 4.4. (4)]. Thus starting with any probability measure $\mu_0$ with compact support, the sequence $\mu_n := M^n(\mu_0)$ converges weakly to $\mu$, i.e. $\mu_n(E) \to \mu(E)$ for all Borel sets $E$. Moreover,

$$\mu = \sum_{i=1}^{r} p_i \cdot w_{i*}\mu, \tag{2.10}$$

which is the only fact concerning $M$ which will actually be used later. For a final remark consider the deterministic algorithm presented above and choose $\mu_0 = \delta_{\{x\}}$ (Dirac measure at $x$). This results in $\mu_n = \sum_{\underline{i} \in I_n} p_{\underline{i}} \cdot \delta_{w_{\underline{i}}(x)}$, provided that the contractions $w_1, \ldots, w_r$ are injective. *If in addition the sets $w_i(O)$ are mutually disjoint and the maps $w_i$ are open, then for any word $\underline{i}$*

$$\mu(V_{\underline{i}}) = p_{\underline{i}}. \tag{2.11}$$

For a proof choose $x \in O$ and take a word $\underline{j}$ with $|\underline{j}| \geq |\underline{i}|$. If $\underline{j} \# \underline{i}$ (2.5) then $w_{\underline{j}}(O)$ and $w_{\underline{i}}(O)$ are disjoint open sets and hence $w_{\underline{j}}(x)$ is not contained in $V_{\underline{i}}$. If, on the other hand, $\underline{j} = \underline{i} * \underline{k}$ for some word $\underline{k}$, then $w_{\underline{j}}(O)$ is a subset of $w_{\underline{i}}(O)$ and $w_{\underline{j}}(x)$ lies in $V_{\underline{i}}$. For $\mu_n = M^n(\delta_{\{x\}})$ and $n \geq |\underline{i}|$ one obtains $\mu_n(V_{\underline{i}}) = p_{\underline{i}}$.

## 2.3 Spectrum of Self-Similar Multifractals

In many publications the geometric and multifractal properties of self-similar sets are studied. They are of interest for their own as well as for applications. See [Falc4, Bar, Mor, Hut, Bed2, CM, HJKPS, HP, GH1, V1, TV]. There is a simple formula for the singularity exponents of the corresponding CMF, which is already well known. But as far as we are aware of it, a rigorous proof still awaits to be written down. Thus this section is considered an important contribution in this field. It provides a proof of the mentioned formula under slightly more general conditions and an application to self-similar measures.
Let $(w_1, \dots, w_r)$ be a set of contracting *similarities of* $\mathbb{R}^d$ *with ratios* $\lambda_1, \dots, \lambda_r$, i.e. $\lambda_i \in ]0,1[$ and $|w_i(x) - w_i(y)| = \lambda_i \cdot |x - y|$ $\forall x \forall y \in \mathbb{R}^d$ $(i = 1, \dots, r)$. Assume further the existence of a nonempty open bounded set $O$ such that

$$w_i(O) \subset O \quad (i = 1, \dots, r) \qquad \text{and} \qquad w_i(O) \cap w_j(O) = \emptyset \quad (i \neq j). \tag{2.12}$$

This property was termed *open set condition*, or *OSC* for short. $O$ will be called a *basic open set*. Any r-adic Cantor set $K$ constructed using an IFS as above is said to be *self-similar* [Hut]. Moran [Mor] was able to give the box dimension as well as the Hausdorff dimension (see page 52 for a definition) of self-similar sets: they both equal the unique $D$, which solves the equation

$$\sum_{i=1}^{r} \lambda_i^D = 1. \tag{2.13}$$

To obtain this result one has to use the construction of $K$ provided by (2.7): due to the OSC the sets $V_{\underline{i}}$ with fixed $|\underline{i}|$ have the same shape and cannot overlap. This property will also be used in the calculation of the singularity exponents.

**Definition 2.4** *Let $w_1, \dots, w_r$ be similarities of $\mathbb{R}^d$ with ratios $\lambda_i \in ]0,1[$ and such that the OSC holds. Let $(p_1, \dots, p_r)$ be a probability vector. Then the CMF $\mu = \langle w_1, \dots, w_r; p_1, \dots, p_r \rangle$ is called a* Self-similar Multifractal *(for short: SMF)* with ratios $\lambda_1, \dots, \lambda_r$ and probability vector $(p_1, \dots, p_r)$. *It makes sense to use the abbreviation*

$$\mu = \langle\langle \lambda_1, \dots, \lambda_r; p_1, \dots, p_r \rangle\rangle$$

*since the spectrum of $\mu$ is determined by these numbers.*

Note that the distribution of a SMF $\mu$ is under excellent control since (2.11) holds. To compute the spectrum of a SMF one could deduce a recursive law for $S_\delta(q)$ from the invariance of $\mu$ (2.10), similar as it is done to obtain the box dimension of $K$ [Bed3, BEH, BEHM, Ma2, Ma1]. The intuitive argument is the following [HP]:
The sets $G_\delta$ are split into $G_\delta(i) := \{B \in G_\delta \,:\, B \subset V_i\}$ $(i = 1, \dots, r)$. The invariance of $\mu$ then supports the approximation

$$S_\delta(q) \simeq \sum_{i=1}^{r} \sum_{B' \in G_\delta(i)} \mu((B')_1)^q \simeq \sum_{i=1}^{r} \sum_{B \in G_{\delta/\lambda_i}} (p_i \mu((B)_1))^q = \sum_{i=1}^{r} p_i^q S_{\delta/\lambda_i}(q).$$



The boundedness $c^{-1} \leq S_\delta(q)\delta^\gamma \leq c$ for $\delta \in$ $[\overline{\lambda}, 1]$ (2.18) extends to $[\overline{\lambda}^2, 1]$ by

$$S_\delta(q)\delta^\gamma \simeq \sum_{i=1}^{r} p_i^q \lambda_i^\gamma \left( S_{\delta/\lambda_i}(q)(\delta/\lambda_i)^\gamma \right),$$

and inductively to $]0,1]$, provided $\gamma$ is chosen such that

$$\sum_{i=1}^{r} p_i^q \lambda_i^\gamma = 1. \tag{2.14}$$

Thus $T(q)$ must equal the unique solution of (2.14). By corollary 2.1 this formula is a generalization of (2.13).
The calculation above is certainly valid when the similarities $w_i$ respect some mesh (i.e. when $\delta_n$-boxes are mapped onto $\delta_{n+1}$-boxes for all $\delta_n$ of an admissible sequence). This is indeed true for the middle third Cantor set. But in general, serious difficulties seem to arise when one is obliged to estimate not only $S_\delta(0)$ (a simple counting task leading to $d_{\text{box}}(K)$) but to treat $S_\delta(q)$ for $q \neq 0$. In particular it is troublesome to establish rigorously the recursion sketched above. It seems that this was not recognized in [HP].
However, we prefer a different approach: We compare the covering by boxes $B$ from $G_\delta$ with the covering by cylindrical sets $V_{\underline{i}}$ with $\underline{i}$ from $J_\delta$—with the approximation

$$S_\delta(q)\delta^\gamma = \sum_{B \in G_\delta} \mu((B)_1)^q \delta^\gamma \simeq \sum_{\underline{i} \in J_\delta} \mu(V_{\underline{i}})^q \delta^\gamma \simeq \sum_{\underline{i} \in J_\delta} p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^\gamma$$

in mind. When $\gamma$ is chosen according to (2.14) the last sum equals exactly 1 for all $\delta$, and the value of $T$ is determined. This procedure has the advantage of not using the maps $w_i$. Only some control about the shape of $V_{\underline{i}}$ and about the possible extent of overlapping is needed. Thus the obtained result is valid for multifractals arising from a more general construction than SMFs, such as example 2.1. As a further condition the intersection of $V_{\underline{i}}$ and $K$ must not be unnaturally small. This is necessary for the same reason which brought us to work with $\mu(B)_1$ of boxes with $\mu(B) \neq 0$. Consequently, it is only needed in order to deal with negative $q$.
Denoting the open ball with center $a$ and radius $\rho$ by $U(a, \rho)$ this reads as follows:

**Theorem 2.3** *Let $\mu = \langle \pi; p_1, \dots, p_r \rangle$ be a CMF, let $\rho_2 > \rho_1 > 0$ and let $\lambda_1, \dots, \lambda_r$ be numbers from $]0,1[$ such that for every word $\underline{i} \in I$ there is a point $x_{\underline{i}}$ in $V_{\underline{i}}$ with*

$$U(x_{\underline{i}}, 2\rho_1 \lambda_{\underline{i}}) \subset V_{\underline{i}} \subset U(x_{\underline{i}}, \rho_2 \lambda_{\underline{i}}), \tag{2.15}$$

$$U(x_{\underline{i}}, 2\rho_1 \lambda_{\underline{i}}) \cap V_{\underline{j}} = \emptyset \quad \text{for all } \underline{j} \neq \underline{i} \text{ with } |\underline{i}| = |\underline{j}|, \tag{2.16}$$

$$\mu(U(x_{\underline{i}}, \rho_1 \lambda_{\underline{i}})) \neq 0. \tag{2.17}$$

*Then $T(q)$ is grid-regular for all $q \in \mathbb{R}$ and equals the unique solution $\gamma$ of (2.14).*

**Proof**

**o)** The functions $x \mapsto p_i^q \lambda_i^x$ are strictly monotonous decreasing with range $\mathbb{R}^+$. Thus (2.14) has exactly one solution, which will be denoted by $\gamma$. Also for later use set

$$\underline{\lambda} := \min\{\lambda_1,\dots,\lambda_r\} \qquad \overline{\lambda} := \max\{\lambda_1,\dots,\lambda_r\}. \tag{2.18}$$

Note two properties of $J_\delta$, which follow immediately from its construction. From (2.5) and (2.6):

$$\sum_{\underline{i}\in J_\delta} p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^\gamma = 1. \tag{2.19}$$

From (2.16) and $V_{\underline{j}} \subset V_{j_1\dots j_n}$ $(n \le |\underline{j}|)$:

$$U(x_{\underline{i}}, 2\rho_1\lambda_{\underline{i}}) \cap U(x_{\underline{j}}, 2\rho_1\lambda_{\underline{j}}) = \emptyset \quad \text{for all } \underline{i}\#\underline{j}, \tag{2.20}$$

in particular for all $\underline{i}\ne\underline{j}$ from $J_\delta$ by (2.5).

Throughout the proof the assumption $\delta \in ]0,\underline{\lambda}[$ is made. First let $q \ge 0$.

**i)** Take $B\in G_\delta$. For the sake of shortness write $J_\delta(E) := \{\underline{i}\in J_\delta : V_{\underline{i}}\cap E \ne\emptyset\}$. It is an important fact that $\#J_\delta((B)_1)$ is bounded by a number which depends neither on $\delta$ nor on $B$. To establish this bound it is enough to have numbers $\rho_4>\rho_3>0$ such that the intersecting sets $V_{\underline{i}}$ have diameters bounded by $\rho_4\cdot\mathrm{diam}((B)_1)$ and contain mutually disjoint balls with radius greater than $\rho_3 \cdot \mathrm{diam}((B)_1)$: just remark that

$$\#J_\delta((B)_1) \le b_1 := \frac{\mathrm{Vol}(U(0,1+\rho_4))}{\mathrm{Vol}(U(0,\rho_3))}. \tag{2.21}$$

But, due to (2.15), (2.20) and (2.4), one may choose $\rho_4 = 2\rho_2(3\sqrt{d})^{-1}$ and $\rho_3 = 4\rho_1\underline{\lambda}(3\sqrt{d})^{-1}$. Now let us proceed to the estimation of $S_\delta(q)$ from above. Since $J_\delta$ is secure (2.6)

$$\mu((B)_1)^q \le \Big(\sum_{J_\delta((B)_1)} \mu(V_{\underline{i}})\Big)^q \le \Big(b_1\cdot \max_{J_\delta((B)_1)}\mu(V_{\underline{i}})\Big)^q \le b_1{}^q\cdot \sum_{J_\delta((B)_1)}\mu(V_{\underline{i}})^q.$$

Taking the sum over all $B\in G_\delta$ yields:

$$S_\delta(q) = \sum_{B\in G_\delta}\mu((B)_1)^q \le b_1{}^q \sum_{B\in G_\delta}\sum_{\underline{i}\in J_\delta((B)_1)} \mu(V_{\underline{i}})^q \le b_1{}^q b_2 \sum_{\underline{i}\in J_\delta}\mu(V_{\underline{i}})^q.$$

Here the constant $b_2$ is obtained from (2.21) by interchanging the roles of $(B)_1$ and $V_{\underline{i}}$ and by setting $\rho_4 = 3\sqrt{d}(4\rho_1\underline{\lambda})^{-1}$ and $\rho_3 = (2\rho_2)^{-1}$, yielding for every word $\underline{i}\in J_\delta$:

$$\#\{B\in G_\delta : V_{\underline{i}}\cap(B)_1\ne\emptyset\} = \#\{B\in G_\delta : \underline{i}\in J_\delta((B)_1)\} \le b_2.$$



Finally $\mu(V_{\underline{j}})$ must be compared with $p_{\underline{j}}$. This is trivial for SMFs, but in general these two numbers are not equal. Take $\underline{j}\in J_\delta$. First $\pi^{-1}(V_{\underline{j}})$ is estimated: assume $\pi(\underline{k}_\infty)\in V_{\underline{j}}$. Since $J_\delta$ is secure (2.6) there is an integer $n$ and a word $\underline{i}\in J_\delta$ with $\underline{i} = (\underline{k}_\infty|n)$. Hence $\pi(\underline{k}_\infty)\in V_{\underline{i}}\cap V_{\underline{j}}$,

$$\pi^{-1}(V_{\underline{j}}) \subset \{\underline{k}_\infty \in I_\infty : \exists n\in\mathbb{N} \text{ with } (\underline{k}_\infty|n)\in J_\delta(V_{\underline{j}})\}$$

and

$$\mu(V_{\underline{j}}) = P[\pi^{-1}(V_{\underline{j}})] \le \sum_{\underline{i}\in J_\delta(V_{\underline{j}})} p_{\underline{i}}.$$

By replacing $(B)_1$ with $V_{\underline{j}}$ and by setting $\rho_4 = \rho_2(2\rho_1\underline{\lambda})^{-1}$, $\rho_3 = 2\rho_1\underline{\lambda}\rho_2^{-1}$ (2.21) provides a constant $b_3$ with

$$\#J_\delta(V_{\underline{j}})\le b_3.$$

Consequently

$$\mu(V_{\underline{j}})^q \le (b_3\cdot\max_{J_\delta(V_{\underline{j}})} p_{\underline{i}})^q \le b_3{}^q \sum_{\underline{i}\in J_\delta(V_{\underline{j}})} p_{\underline{i}}{}^q.$$

Similar as above one obtains

$$\sum_{\underline{j}\in J_\delta}\mu(V_{\underline{j}})^q \le b_3{}^q\sum_{\underline{j}\in J_\delta}\sum_{\underline{i}\in J_\delta(V_{\underline{j}})} p_{\underline{i}}{}^q \le b_3{}^q b_3 \sum_{\underline{i}\in J_\delta} p_{\underline{i}}{}^q.$$

Summarizing,

$$S_\delta(q)\delta^\gamma \le b_1{}^q b_2 b_3{}^{q+1}\sum_{\underline{i}\in J_\delta} p_{\underline{i}}{}^q\delta^\gamma \le b_1{}^q b_2 b_3{}^{q+1} c_1 \sum_{\underline{i}\in J_\delta} p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^\gamma = b_1{}^q b_2 b_3{}^{q+1} c_1.$$

Note that $b_1$, $b_2$, $b_3$ and $c_1 = \max\{1,\underline{\lambda}^{-\gamma}\}$ do not depend on $\delta$.

**ii)** Now $S_\delta(q)$ will be estimated from below. Take $\underline{i}\in J_{\delta'}$ where $\delta' = (3\rho_2)^{-1}\delta$. From $0\ne p_{\underline{i}}\le\mu(V_{\underline{i}})$ follows the existence of a box $B_{\underline{i}}\in G_\delta$ which meets $V_{\underline{i}}$. Since $\mathrm{diam}(V_{\underline{i}}) \le 2\rho_2\lambda_{\underline{i}} < \delta$, the parallel body $(B_{\underline{i}})_1$ contains $V_{\underline{i}}$ and thus $p_{\underline{i}}\le\mu((B_{\underline{i}})_1)$. Moreover, any fixed $\delta$-box can meet at the most $b_4$ (2.21) sets $V_{\underline{i}}$ with $\underline{i}\in J_{\delta'}$. Hence

$$\sum_{\underline{i}\in J_{\delta'}} p_{\underline{i}}{}^q \le b_4\sum_{B\in G_\delta}\mu((B)_1)^q$$

and by (2.4)

$$S_\delta(q)\delta^\gamma \ge b_4^{-1}\sum_{\underline{i}\in J_{\delta'}} p_{\underline{i}}{}^q\delta^\gamma \ge b_4^{-1}c_2\sum_{\underline{i}\in J_{\delta'}} p_{\underline{i}}{}^q\lambda_{\underline{i}}{}^\gamma = \frac{c_2}{b_4},$$

where $c_2 = (3\rho_2)^\gamma\cdot\max\{1,\underline{\lambda}^{-\gamma}\}$ is independent of $\delta$.

**iii)** From i) and ii) follows immediately

$$\gamma + \frac{\log c_2 b_4^{-1}}{-\log\delta} \le \frac{\log S_\delta(q)}{-\log\delta} \le \gamma + \frac{\log b_1{}^q b_2 b_3{}^{q+1}}{-\log\delta}$$

for all sufficiently small $\delta$, i.e. $T(q)$ is grid-regular and equals $\gamma$.

Now let $q < 0$.

**iv)** Again let us give first an upper bound of $S_\delta(q)$. Take $B \in G_\delta$. Set $\delta' = (3\rho_2)^{-1}\delta$. To the contrary with i) a $V_{\underline{i}}$ has to be found with smaller measure than $(B)_1$. This is indeed the easier task since $\mu(B) \neq 0$ due to our definition of $T(q)$. By (2.6) there is $\underline{i} \in J_{\delta'}$ such that $V_{\underline{i}}$ meets $B$. As in ii) $V_{\underline{i}}$ is a subset of $(B)_1$ and thus $0 \neq p_{\underline{i}} \leq \mu((B)_1)$. So

$$S_\delta(q) = \sum_{B \in G_\delta} \mu((B)_1)^q \leq b_5 \sum_{\underline{i} \in J_{\delta'}} p_{\underline{i}}{}^q$$

since any fixed $V_{\underline{i}}$ ($\underline{i} \in J_{\delta'}$) meets at the most some constant number $b_5$ (2.21) of boxes from $G_\delta$. With $c_e = (3\rho_2)^\gamma \cdot \max\{1, \underline{\lambda}^{-\gamma}\}$

$$S_\delta(q)\delta^\gamma \leq b_5 c_3 \sum_{\underline{i} \in J_{\delta'}} p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^\gamma = b_5 c_3.$$

**v)** Take $\underline{i} \in J_{\delta''}$ with $\delta'' = 3\sqrt{d}(\rho_1 \underline{\lambda})^{-1}\delta$. Only here the precondition (2.17) is used, which implies the existence of a box $B(\underline{i}) \in G_\delta$ which meets $U(x_{\underline{i}}, \rho_1\lambda_{\underline{i}})$. Since $\operatorname{diam}(B(\underline{i}))_1 = \rho_1 \underline{\lambda} \delta'' \leq \rho_1 \lambda_{\underline{i}}$ by (2.4),

$$(B(\underline{i}))_1 \subset U(x_{\underline{i}}, 2\rho_1\lambda_{\underline{i}}) \subset V_{\underline{i}}.$$

The first idea now is to continue with $\mu((B(\underline{i}))_1)^q \geq \mu(V_{\underline{i}})^q$. But here the method of i) cannot be applied to compare $\mu(V_{\underline{i}})$ with $p_{\underline{i}}$ 'in the average': $\mu((B(\underline{i}))_1)$ does not have to be bounded by just some $p_{\underline{j}}$, but with $p_{\underline{i}}$ itself. However, this is possible since

$$\pi^{-1}(U(x_{\underline{i}}, 2\rho_1\lambda_{\underline{i}})) \subset \{\underline{k}_\infty \in I_\infty \,:\, (\underline{k}_\infty |\, |\underline{i}|) = \underline{i}\}$$

by (2.16), leading to

$$0 \neq \mu((B(\underline{i}))_1) \leq \mu(U(x_{\underline{i}}, 2\rho_1\lambda_{\underline{i}})) \leq p_{\underline{i}}.$$

This time (2.20) is used instead of (2.21) to conclude that $B(\underline{i}) \neq B(\underline{j})$ for $\underline{i}\#\underline{j}$. From this

$$\sum_{\underline{i} \in J_{\delta''}} p_{\underline{i}}{}^q \leq \sum_{B \in G_\delta} \mu((B)_1)^q = S_\delta(q)$$

and

$$S_\delta(q)\delta^\gamma \geq c_4 \sum_{\underline{i} \in J_{\delta''}} p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^\gamma = c_4.$$

**vi)** From iv) and v) follows the assertion of the theorem for negative $q$. ◇

A first and almost immediate application of theorem 2.3 is the one to CMFs on the real axis $\mathbb{R}$:



**Corollary 2.4** *Let $\mu$ be a CMF on $\mathbb{R}$ such that for all $n \in \mathbb{N}$ the cylindrical sets $V_{\underline{i}}$ ($\underline{i} \in I_n$) are intervals with mutually disjoint interiors, and such that*

$$\operatorname{diam}(V_{\underline{i}}) = \lambda_{\underline{i}} \cdot \operatorname{diam}(V_{\text{nil}})$$

*for all words $\underline{i}$. Then $T(q)$ is grid-regular for all $q \in \mathbb{R}$ and equals the unique solution $\gamma$ of (2.14).*

**Proof** Set $c := \operatorname{diam}(V_{\text{nil}})$ and $\underline{\lambda}$, $\overline{\lambda}$ according to (2.18). Take an arbitrary word $\underline{i}$. Among $V_{\underline{i}*1*1}, \ldots, V_{\underline{i}*r*r}$ there is at least one—say $V_{\underline{i}*j_1j_2}$—with distance at least $\underline{\lambda}^2\lambda_{\underline{i}}c$ from the boundary of $V_{\underline{i}}$. Choose $m$ large enough to ensure $\overline{\lambda}^m < 1/2 \cdot \underline{\lambda}^2$ and set $\underline{j} := j_1j_2 * 1 \ldots 1 \in I_m$. Then $V_{\underline{i}*\underline{j}} \subset V_{\underline{i}*j_1j_2}$ and $\operatorname{diam}(V_{\underline{i}*\underline{j}}) \leq \overline{\lambda}^m \lambda_{\underline{i}} c < 1/2 \cdot \underline{\lambda}^2 \lambda_{\underline{i}} c$. Thus it is enough to choose $x_{\underline{i}} \in V_{\underline{i}*\underline{j}}$ and $\rho_2 = c$, $\rho_1 = c/2 \cdot \underline{\lambda}^2$, since $V_{\underline{i}*\underline{j}} \subset U(x_{\underline{i}}, \rho_1\lambda_{\underline{i}})$, $p_{\underline{i}*\underline{j}} \neq 0$ and $\operatorname{dist}(x_{\underline{i}}, \partial V_{\underline{i}}) \geq 2\rho_1\lambda_{\underline{i}}$. ◇

With the necessary care theorem 2.3 provides the singularity exponents of self-similar measures: The first two preconditions are obviously satisfied due to the OSC. Moreover, also the third one can be verified provided that there is a basic open set $O$ with $O \cap K \neq \emptyset$. This condition was termed *strong OSC*, for short *SOSC* [BG].

**Lemma 2.5** *Let $\mu = \langle\langle \lambda_1, \ldots, \lambda_r; p_1, \ldots, p_r \rangle\rangle$ be any SMF. Then the conclusion of theorem 2.3 holds for $q \geq 0$. If, in addition, the SOSC holds, then this is even true for $q \in \mathbb{R}$.*

**Remark** A simple calculation even shows that the SOSC implies $\mu(O) = 1$ and $\mu(\partial O) = 0$. But still $K$ need not be a subset of $O$.

**Proof** The proof of theorem 2.3 reveals that precondition (2.17) is only needed for negative $q$. So it only remains to verify (2.15)-(2.17) under the assumption $K \cap O \neq \emptyset$. Take $x = \pi(\underline{i}_\infty)$ lying in $O$. Since $O$ is open and bounded there is $\rho_2 > \rho_1 > 0$ such that $U(x, 2\rho_1) \subset O \subset \overline{O} \subset U(x, \rho_2)$ and an integer $n$ such that $\overline{\lambda}^n \cdot \operatorname{diam}(O) \leq \rho_1$. For $\underline{j} = (\underline{i}_\infty | n)$ the set $V_{\underline{j}}$ contains $x$, has diameter $\lambda_{\underline{j}} \cdot \operatorname{diam}(O)$ and is thus a subset of $U(x, \rho_1)$. Letting $x_{\underline{k}} := w_{\underline{k}}(x)$ for all finite words $\underline{k}$ one finds

$$V_{\underline{k}*\underline{j}} = w_{\underline{k}}(V_{\underline{j}}) \subset w_{\underline{k}}(U(x, \rho_1)) = U(x_{\underline{k}}, \rho_1\lambda_{\underline{k}}),$$

hence $\mu(U(x_{\underline{k}}, \rho_1\lambda_{\underline{k}})) \geq p_{\underline{k}*\underline{j}} \neq 0$ and (2.17) is established. Note that

$$U(x_{\underline{k}}, 2\rho_1\lambda_{\underline{k}}) \cap V_{\underline{i}} \subset w_{\underline{k}}(O) \cap w_{\underline{i}}(\overline{O}) = \emptyset$$

for all $\underline{k}\#\underline{i}$, giving (2.16) for this choice of $x_{\underline{i}}$. (2.15) is evident. ◇

Two simple examples of SMFs with the SOSC are the following:

**Example 2.2 (A Class of Totally Disconnected SMFs)** Take a self-similar set $K = \langle w_1, \ldots, w_r \rangle$ and assume that the sets $w_i(K)$ ($i = 1, \ldots, r$) are mutually disjoint. Then $K$ is totally disconnected. Moreover, the union $O$ of all $U(x, \varepsilon)$ with center $x$ in $K$ satisfies the OSC for $\varepsilon < 1/3 \cdot \inf\{\operatorname{dist}(w_i(K), w_j(K)) \,:\, i \neq j\}$, due to the invariance (2.8) of $K$. $O$ even contains $K$. ○

**Example 2.3 (SMFs on the Real Axis)** For SMFs on the real axis ($d = 1$) the situation is even trivial: $K$ is uncountable provided $r \geq 2$ and since the boundary of any open subset of $\mathbb{R}$ is countable, $K$ must intersect $O$. Note that there are SMFs on $\mathbb{R}$ with infinitely connected basic open set. Corollary 2.4 cannot be applied then. ○

At the time when this thesis was submitted, it was not clear whether the OSC implies the SOSC in general. The state of knowledge was the comparison of the two conditions by [BG]. Therefore it was opportune to present a geometrical situation in which the OSC does imply the SOSC. Two examples shall illuminate the problem first.

**Example 2.4 (K Lying on the Fractal Boundary of O)** Let $r = 4$ and

$$w_i(x) := 1/3 \cdot (x - a_i) + a_i$$

with $a_1 = (0,0)$, $a_2 = (1,0)$, $a_3 = (1,1)$ and $a_4 = (0,1)$. Set $O := ]0,1[^2 \backslash K$ where $K = \langle w_1, \ldots, w_4 \rangle$ as usual. $O$ is of course open and bounded and $K$ lies on its boundary. As will be shown it is even a basic open set for $(w_1, \ldots, w_4)$.
Take $j \in \{1, \ldots, 4\}$ and assume there is a point $x \in O$ such that $w_j(x) \in K$. Of course $w_j(x) \in w_j(\overline{O})$. Since $w_i(K) \subset w_i(\overline{O})$ $(i = 1, \ldots, 4)$, which are mutually disjoint in this example, $w_j(x)$ must by (2.8) be contained in $w_j(K)$. The bijectivity of $w_j$ now implies $x \in K$, which is a contradiction to $x \in O$. Thus $w_j(O)$ is a subset of $]0,1[^2 \backslash K = O$ for $(j = 1, \ldots, 4)$. Concerning the disjointness: $w_i(O) \cap w_j(O) \subset w_i(]0,1[^2) \cap w_j(]0,1[^2) = \emptyset$, provided $i \neq j$. This proves the claim. ○

**Example 2.5 (K Lying on the Smooth Boundary of O)** Take $d \geq 2$ , $r = 2$ and

$$w_1(x) := 1/3 \cdot x, \quad w_2(x) := 1/3 \cdot x + (2/3, 0, \ldots, 0).$$

Let $C$ denote the middle third Cantor set on $\mathbb{R}$ (see Ex. 1.1). Then $K = \langle w_1, w_2 \rangle = \{(x, 0, \ldots, 0) : x \in C\}$, since this set is compact and invariant. Obviously, $K$ lies on the boundary of the basic open set $O := ]0,1[^d$. ○

Thus it is quite possible that $K$ has no point in common with a particular basic open set $O$. But the above examples support the intuition that when this happens, either $O$ has a highly irregular boundary, or the dimension of the embedding space $\mathbb{R}^d$ has been chosen too large. Moreover, a better chosen $O$ satisfies the SOSC. In applications it is often possible to find a basic open set $O \subset \mathbb{R}^d$ with quite regular boundary: $O$ is the union of a finite set of polyeders, i.e. its boundary $\partial O$ lies on a finite union of $(d-1)$-dimensional hyperplanes. We shall say then that $\partial O$ is *piecewise linear*. Our aim is to prove that such a basic open set implies the SOSC und is thus sufficient for the determination of $T$.



**Lemma 2.6** *Let $(w_1, \ldots, w_r)$ be a set of contracting similarities with basic open set $O$, the boundary $\partial O$ of which is piecewise linear. Then the SOSC holds with some set $O^*$.*

**Proof** The case $r = 1$ is trivial, thus assume $r \geq 2$.

**i)** Let $H$ be the linear subspace spanned by $K$ and denote its dimension by $d'$. Due to its minimality $H'$ is invariant under $w_i$: choose $d' + 1$ points of $K$ which span $H'$. By the invariance of $K$ (2.8) their images under $w_i$ must again be contained in $K$ which is a subset of $H$. They span a hyperplane of dimension $d'$. Thus $w_i(H) = H$ $(i = 1, \ldots, r)$.

**ii)** Now it will be proved that the SOSC holds for $K$ as a subset of $H$. Denote the interior of $H \cap \overline{O}$ with respect to $H$ by $O'$. Note that $K$ is contained in the closure $\overline{O'}^H$ of $O'$ with respect to $H$, since $K \subset \overline{O}$ and since $O'$ is perfect. We claim that $O'$ and $K$ have a point in common. Assume the contrary. As a subset of $\partial_H O'$, $K$ must be contained in a finite union of linear subspaces $E_k$ $(k = 1, \ldots, N)$ of dimension $d'' = d' - 1$. As it will be shown, $K$ lies then in a linear subspace of dimension less or equal to $d'' < d'$, in contradiction to the definition of $H$.
Assume first a $j$ and a point $x$ of $K$ such that $x$ lies in $E_j$, but in no $E_k$ with $k \neq j$. Since $x$ has positive distance to $E_k$ $(k \neq j)$, there is a finite word $\underline{i}$ such that $V_{\underline{i}}$ contains $x$ but meets no $E_k$ $(k \neq j)$. From

$$w_{\underline{i}}(K) \subset V_{\underline{i}} \cap K \subset E_1$$

follows that $K$ lies in $w_{\underline{i}}^{-1}(E_1)$, a $d''$-dimensional linear subspace. Otherwise, i.e. if there is no such $x$, $K$ must be a subset of the union of all $E_{k_0} \cap E_{k_1}$, where multiply occurring sets have been removed. Inductively the same argumentation as just given can be applied or $K$ can be found to be contained in $E_{k_0} \cap \ldots \cap E_{k_{d'}}$. This is a finite union of points, in contradiction to $r \geq 2$.

**iii)** The desired basic open set $O^*$, which intersects $K$, is now readily constructed. Denoting the component of $y \in \mathbb{R}^d$ in $H$ by $h$ and the one perpendicular to $H$ by $h^\perp$, set

$$O^* := \{y \in \mathbb{R}^d : h \in O' \text{ and } |h^\perp| < 1\}.$$

(Compare Ex. 2.5.) Certainly $O^*$ is bounded and open, and $K \cap O^* = K \cap O' \neq \emptyset$. Due to the shape of $O$ and due to the invariance of $H$ the OSC of $O$ carries over to $O'$ as a subset of $H$. Since $H^\perp$ is invariant as well this establishes $O^*$ as a basic open set. ◇

Summarizing, the author was able to give three conditions, each sufficient to imply the SOSC: $d = 1$, the sets $w_i(K)$ are mutually disjoint, or $\partial O$ is piecewise linear. However, after this thesis was submitted the author's attention was brought to [Sch], due to which the SOSC holds for any SMF.

**Corollary 2.5** *Let $\mu = \langle\langle \lambda_1, \ldots, \lambda_r; p_1, \ldots, p_r \rangle\rangle$ be a SMF in $\mathbb{R}^d$. Then $T(q)$ is grid-regular for all $q \in \mathbb{R}$ and uniquely determined by the equation*

$$\sum_{i=1}^{r} p_i^q \lambda_i^{T(q)} = 1. \tag{2.22}$$

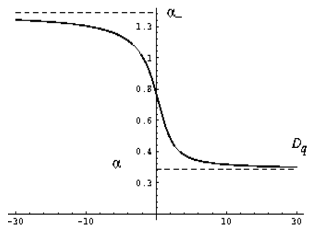


Figure 2.3: *The typical feature of the generalized dimensions $D_q$ of a SMF, plotted as a function of $q$. Here $r = 4$, $\lambda_1 = \ldots = \lambda_4 = 1/6$, $p_1 = p_2 = 1/10$, $p_3 = 1/5$ and $p_4 = 3/5$.*

Formulas related to (2.22) have been found previously, but with less generality: In [HJKPS] differentiability of the spectrum $f$ is assumed. In [HP] the formula is derived only for positive $q$ and the argumentation is only valid when the similarities respect some mesh (compare page 40).
From (2.22) the spectrum $F$ follows immediately by an application of theorem 1.2. To give as much information as possible we set:

$$\alpha_\infty := \min_{i=1,\ldots,r} \frac{\log p_i}{\log \lambda_i} \quad \alpha_1 := \frac{\sum_{i=1}^{r} p_i \log p_i}{\sum_{i=1}^{r} p_i \log \lambda_i} \quad \alpha_0 := \frac{\sum_{i=1}^{r} \lambda_i^D \log p_i}{\sum_{i=1}^{r} \lambda_i^D \log \lambda_i} \quad \alpha_{-\infty} := \max_{i=1,\ldots,r} \frac{\log p_i}{\log \lambda_i},$$

where $D$ is the box dimension of $K = \text{supp}(\mu)$ and satisfies (2.13). Thereby the various values of $\alpha$ have interpretations as particular 'local Hölder exponents': $\alpha_\infty$ and $\alpha_{-\infty}$ as the Hölder of the most probable and the most rarefied points, respectively, and $\alpha_1$ and $a_0$ as the Hölder occurring most probably with respect to the underlying measure $\mu$ and the $D$-dimensional Hausdorff measure, respectively. While the first two interpretations follow from the theorem below, see (2.25) or [EM] for the other two.
Furthermore, we denote by $\gamma_1$, and $\gamma_2$ the unique solutions of the equations

$$\sum_{p_i = \lambda_i^{\alpha_\infty}} \lambda_i^{\gamma_1} = 1 \quad \text{resp.} \quad \sum_{p_i = \lambda_i^{\alpha_{-\infty}}} \lambda_i^{\gamma_2} = 1,$$

where the sums are taken over all numbers $i \in \{1, \ldots, r\}$ which satisfy the indicated condition.



**Theorem 2.6 (Spectrum of SMFs)** *Let $\mu$ be a CMF, for which (2.22) holds for all real $q$. Then*

$$F(\alpha) = \inf_{q \in \mathbb{R}} (T(q) + q\alpha) = \begin{cases} -\infty & \alpha < \alpha_\infty \\ \gamma_1 & \alpha = \alpha_\infty \\ T(q) - qT'(q) & \alpha = -T'(q) \in ]\alpha_\infty, \alpha_{-\infty}[ \\ \gamma_2 & \alpha = \alpha_{-\infty} \\ -\infty & \alpha > \alpha_{-\infty}, \end{cases}$$

*and $F$ is grid-regular everywhere provided $T$ is. Moreover, $F(\alpha)$ is continuous and strictly concave in $[\alpha_\infty, \alpha_{-\infty}]$ and $C^\infty$ in $]\alpha_\infty, \alpha_{-\infty}[$. Its graph touches the internal bisector of the axes at $\alpha_1$ and attains the maximal value $D$ at $\alpha_0$. Furthermore, the $2 \times 2$ equation system*

$$\left| \begin{array}{lcl} \sum_{i=1}^{r} \left(\frac{p_i}{\lambda_i^\alpha}\right)^q \cdot \lambda_i^\gamma & = & 1 \quad \text{(a)} \\ \sum_{i=1}^{r} \log\left(\frac{p_i}{\lambda_i^\alpha}\right) \left(\frac{p_i}{\lambda_i^\alpha}\right)^q \cdot \lambda_i^\gamma & = & 0 \quad \text{(b)} \end{array} \right| \tag{2.23}$$

*is for every $\alpha \in ]\alpha_\infty, \alpha_{-\infty}[$ uniquely solved by $\gamma = F(\alpha)$, $q = F'(\alpha)$.*

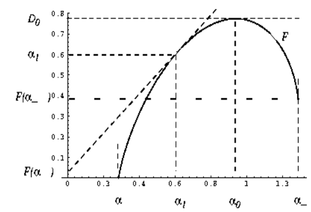


Figure 2.4: *The typical feature of the spectrum $F(\alpha)$ of a SMF. Here $r = 4$, $\lambda_1 = \ldots = \lambda_4 = 1/6$, $p_1 = p_2 = 1/10$, $p_3 = 1/5$ and $p_4 = 3/5$ as in figure 2.3.*

See figure 2.4 for the feature of a typical spectrum. For an intuitive explanation of the back ground of (2.23) it must be referred to the remark on page 86, where the involved notation is at hand.

**Corollary 2.7** *Let $\mu$ be a CMF, for which (2.22) holds for all real $q$. Then $D_0 = D = \overline{d_{\text{box}}}(K)$, $D_1 = \alpha_1$ and*

$$D_\infty = \lim_{q \to \infty} D_q = \alpha_\infty \leq D_q \leq \alpha_{-\infty} = \lim_{q \to -\infty} D_q = D_{-\infty}.$$

*Moreover, $D_q$ is either constant or strictly decreasing and continuously differentiable.*

**Proof** Write $c_i = c_i(\alpha) = \log(p_i) - \alpha \log(\lambda_i)$ $(i = 1, \ldots, r)$ for short. The $c_i$ are strictly increasing functions of $\alpha$ with zeros $\log p_i / \log \lambda_i$.

**o)** First the trivial case. When all zeros of the $c_i$ coincide, then $p_i = \lambda_i^D$ where $D$ is the common zero. Necessarily $D = T(0) = d_{\text{box}}(K)$ by (2.22) and corollary 2.1. From (2.22) follows $T(q) = (1-q)D$ and $\alpha_\infty = \alpha_1 = \alpha_0 = \alpha_{-\infty} = \gamma_1 = \gamma_2 = D$. Theorem 1.1 gives $F(\alpha) = T(q) - qT'(q) = D$ for $\alpha = D = -T'(q)$; (2.23) is trivial and $D_q = D$ for all $q$. The grid-regularity as well as $F(\alpha) = -\infty$ ($\alpha \neq D$) follow as in iv) below. In this case there is nothing more to prove.

From now on the case o) is excluded. Equivalently $\alpha_\infty < \alpha_{-\infty}$ may be assumed.

**i)** First it will be shown that (2.23) is solvable exactly if $\alpha$ lies in the range of $-T'$, and that it determines $F$ and $F'$. Assume first that $(\gamma_0, q_0)$ solves the system for some fixed $\alpha$, and rewrite it as:

$$\left| \begin{array}{rcl} \sum_{i=1}^r p_i^{q_0} \lambda_i^{\gamma_0 - \alpha q_0} & = & 1 \\ \sum_{i=1}^r p_i^{q_0} \lambda_i^{\gamma_0 - \alpha q_0} \log p_i & = & \alpha \sum_{i=1}^r p_i^{q_0} \lambda_i^{\gamma_0 - \alpha q_0} \log \lambda_i \end{array} \right|$$

Consequently $T(q_0) = \gamma_0 - \alpha q_0$, $\alpha = -T'(q_0)$ and with theorem 1.1 $F(\alpha) = T(q_0) - q_0 T'(q_0) = \gamma_0$ and $F'(\alpha) = q_0$ is obtained. Thus the solution is even unique. On the other hand, it is now easy to see that, if $\alpha = -T'(q_1)$, then $(T(q_1) + \alpha q_1, q_1)$ provides a solution of (2.23).

**ii)** Now let us determine the range of $-T'$. Instead of using the implicit formula arising from (2.22), which is troublesome to handle, it will be shown that (2.23) is solvable exactly for $\alpha \in ]\alpha_\infty, \alpha_{-\infty}[$. Consider the second equation (2.23.b). For fixed $\gamma$ the function $q \mapsto \sum c_i e^{c_i q} \lambda_i{}^\gamma$ is strictly increasing. For $\alpha \leq \alpha_\infty$ ($\alpha \geq \alpha_{-\infty}$), however, it is strictly negative (positive), since case o) is excluded. Hence (2.23.b) has then no solution. From now on fix $\alpha \in ]\alpha_\infty, \alpha_{-\infty}[$. Then $c_i < 0 < c_j$ for some $i$ and $j$, and a unique solution $q(\gamma)$ of (2.23.b) exists. By the implicit function theorem $q(\gamma)$ depends continuously differentiable on $\gamma$ since $\sum c_i^2 e^{c_i q} \lambda_i^\gamma \neq 0$. Now turn to (2.23.a) and consider the strictly positive function $h(\gamma) = \sum e^{c_i q(\gamma)} \lambda_i^\gamma$. Its derivate satisfies

$$h'(\gamma) = \sum_{i=1}^r e^{c_i q(\gamma)} \lambda_i^\gamma \log(\lambda_i) + \sum_{i=1}^r e^{c_i q(\gamma)} \lambda_i^\gamma c_i q'(\gamma) \leq \log \overline{\lambda} \cdot h(\gamma) < 0,$$

since the second term vanishes by definition of $q(\gamma)$. So, $h$ is strictly decreasing and the mean value theorem of calculus implies for $\gamma < D$:

$$h(\gamma) - h(D) = (D - \gamma)(-h'(x_\gamma)) \geq \log(1/\overline{\lambda}) \cdot (D - \gamma) h(D).$$

Thus $h(\gamma) \to \infty$ ($\gamma \to -\infty$). On the other hand,

$$h(D) = \sum_{i=1}^r e^{c_i q(D)} \lambda_i^D \leq \sum_{i=1}^r e^{c_i \cdot 0} \lambda_i^D = 1$$

because $q(\gamma)$ minimizes by its definition the strictly convex function $q \mapsto \sum e^{c_i q} \lambda_i^\gamma$ ($\gamma$ fixed). Summarizing there is a unique $\gamma$ with $h(\gamma) = 1$ and existence and uniqueness of the solution of (2.23) is established. Moreover, $\gamma \leq D$, hence $F(\alpha) \leq D$.



**iii)** As a consequence of i) and ii) the range of $-T'$ is exactly $]\alpha_\infty, \alpha_{-\infty}[$. By implicit differentiation of (2.22) it is easy to derive the strict convexity of $T$. As a consequence $D_q$ is strictly decreasing, which is a generalization of the same result for positive $q$ in [HP]. Furthermore,

$$\lim_{q \to \pm\infty} D_q = \lim_{q \to \pm\infty} -T'(q) = a_{\pm\infty}.$$

Concerning differentiability of $D_q$ at $q = 1$: Proposition 1.19 applies and $D_1 = -T'(1) = \alpha_1$. For $q \neq 1$

$$\frac{d}{dq} D_q = \frac{(1-q)T'(q) + T(q)}{(1-q)^2} = (1-q)^{-2} (F(\alpha) - \alpha)\big|_{\alpha = -T'(q)}$$

which converges to $-T''(1)/2$ as $q \to 1$. Since $D_q$ is continuous at 1, it is hence also continuously differentiable there. The last equation reveals in addition that the touching of $F$ and the inner bisector of the axis is of order two, provided $q$ is taken as the curve parameter.

**iv)** Turning to the spectrum proposition 1.15 and iii) show that $F(\alpha) \leq F^m(\alpha) \leq \inf_q T(q) + \alpha q = -\infty$ for $\alpha$ outside $[\alpha_\infty, \alpha_{-\infty}]$. The concavity and differentiability follow from theorem 1.1 and the implicit function theorem. The grid-regularity is a consequence of theorem 1.2. Regarding i) the maximum of $F$ is discovered by noting that $(\gamma, q) = (D, 0)$ solves (2.23) for $\alpha = \alpha_0$.

**v)** So, it remains only to compute the values $F(a_{\pm\infty})$. But by theorem 1.1 $F(\alpha)$ is continuous in $[\alpha_\infty, \alpha_{-\infty}]$. So the behaviour of the solutions of (2.23) has to be studied near $a_{\pm\infty}$. For an easy presentation of the proof assume without loss of generality that

$$\alpha_\infty = \frac{\log p_i}{\log \lambda_i} \qquad \Leftrightarrow \qquad i \in \{1, \ldots, t\}$$

for some $t < r$. We will consider arbitrarily $\alpha \in ]\alpha_\infty, \alpha_\infty + \varepsilon[$ where $\varepsilon > 0$ is chosen small enough to guarantee the existence of $c'' > c' > 0$ such that $c' \leq -c_i(\alpha) \leq c''$ ($i = t+1, \ldots, r$) and $c_k(\alpha) \geq 0$ ($k = 1, \ldots, t$). Denote the solution of (2.23) by $(F(\alpha), q(\alpha))$. Since $F(\alpha) \geq 0$ (2.23.b) implies

$$\sum_{k=1}^t \lambda_k^{F(\alpha)} c_k(\alpha) e^{c_k(\alpha) q(\alpha)} = \sum_{i=t+1}^r \lambda_i^{F(\alpha)} (-c_i(\alpha)) e^{c_i(\alpha) q(\alpha)} \leq r c'' \cdot e^{-c' \cdot q(\alpha)}.$$

The terms in the first sum are all positive, $F(\alpha) \leq D$, and $q(\alpha) \to \infty$ ($\alpha \downarrow \alpha_\infty$) by iii). Thus

$$0 \leq c_k(\alpha) q(\alpha) \leq \text{const} \cdot q(\alpha) e^{-c' \cdot q(\alpha)} e^{-c_k(\alpha) q(\alpha)} \to 0 \quad (\alpha \downarrow \alpha_\infty) \quad (k = 1, \ldots, t).$$

On the other hand, $c_i(\alpha) q(\alpha) \to -\infty$ ($\alpha \downarrow \alpha_\infty$) ($i = 1 + t, \ldots, r$) is trivial. Moreover, for reasons of continuity, $F(\alpha) \to F(\alpha_\infty)$ and with (2.23.a)

$$1 = \sum_{k=1}^t \lambda_k^{F(\alpha)} e^{c_k(\alpha) q(\alpha)} + \sum_{i=t+1}^r \lambda_i^{F(\alpha)} e^{c_i(\alpha) q(\alpha)} \to \sum_{k=1}^t \lambda_k^{F(\alpha_\infty)}.$$

From this $F(\alpha_\infty) = \gamma_1$. A similar argument shows $F(\alpha_{-\infty}) = \gamma_2$. $\diamond$

## 2.4 Examples and Counterexamples

The title of this section speaks for itself. Among other things we treat some easy cases and contribute to the interpretation of $F(\alpha)$ as the dimension of subsets of $K = \operatorname{supp}(\mu)$ with 'local Hölder exponent $\alpha$'. The counterexamples prove the necessity of the preconditions of some of our theorems and show that the spectrum $F$ need not be concave.

### 2.4.1 Homogeneous Multifractals

In this subsection only SMFs $\mu = \mu(p) = \langle\langle\lambda_1, \ldots, \lambda_r; p_1, \ldots, p_r\rangle\rangle$ are considered, where the involved similarities $w_i$ are regarded as fixed and $p = (p_1, \ldots, p_r)$ as variable.
No matter how $p$ is chosen, $K = \operatorname{supp}(\mu) = \langle w_1, \ldots, w_r\rangle$ remains the same self-similar set with $d_{\mathrm{box}}(K) = D$ determined by (2.13). But—assuming that (2.22) holds—the spectrum varies with $p$ getting more and more narrow as $D_\infty \to D_{-\infty}$. Thus the coincidence $D_\infty = D_{-\infty}$ is a special case, which occurs exactly for

$$p_i = p_i^* := \lambda_i^D.$$

In particular $D_q$ is in this case constant and any multifractal with $D_q \equiv D_0$ is called *homogeneous* or *uniform* [HP, HJKPS]. The choice of probabilities $p = p^*$ may be considered to be inappropriate since $\mu(p^*)$ reveals no structure of $K$. But homogeneous multifractals possess some extremal properties and are, therefore, of theoretical interest. To state the mentioned properties of $\mu(p^*)$ and also for further use in this section the following definitions are needed: the *capacity* of a multifractal $\mu$ [Y, page 119]:

$$d_C(\mu) := \sup_{\delta>0} \inf\{d_{\mathrm{box}}(E) \,:\, E \subset K,\, \mu(E) \geq 1-\delta\},$$

the *$\alpha$-dimensional Hausdorff measure* $m^\alpha$ (for details see [Falc4, Rog]):

$$m_\varepsilon^\alpha(E) := \inf\{\sum_{i=1}^\infty (\operatorname{diam}(S_i))^\alpha \,:\, E \subset \bigcup_{i=1}^\infty S_i,\ \operatorname{diam}(S_i) \leq \varepsilon \ \ \forall i \in \mathbb{N}\}$$

$$m^\alpha(E) := \lim_{\varepsilon \downarrow 0} m_\varepsilon^\alpha(E) = \sup_{\varepsilon>0} m_\varepsilon^\alpha(E),$$

the *Hausdorff dimension* $d_{HD}(E)$ of a set $E$:

$$d_{HD}(E) := \inf\{\alpha \geq 0 \,:\, m^\alpha(E) = 0\} = \sup\{\alpha \geq 0 \,:\, m^\alpha(E) = \infty\}$$

and finally the Hausdorff dimension of a multifractal [Y, page 115]:

$$d_{HD}(\mu) := \inf\{d_{HD}(E) \,:\, E \subset K,\, \mu(E) = 1\}.$$

Note that $d_{HD}(E)$ exists, since $m_\varepsilon^\alpha(E) \leq \varepsilon^{\alpha-\beta} m_\varepsilon^\beta(E)$ whenever $\alpha > \beta$, and thus $m^\beta(E) < \infty \Rightarrow m^\alpha(E) = 0$ and $m^\alpha(E) > 0 \Rightarrow m^\beta(E) = \infty$.



**Example 2.6 (Maximality of the Hausdorff Dimension of $\mu(p^*)$)** A result of Geronimo et Hardin [GH1, page 92] reads in our situation as follows:
*Assume that $\mu$ is a SMF and that $w_i(K) \cap w_j(K) = \emptyset$ for all $i \neq j$. Then*

$$d_{HD}(\mu(p)) = d_C(\mu(p)) = D_1 = \frac{\sum_{i=1}^r p_i \log p_i}{\sum_{i=1}^r p_i \log \lambda_i}. \tag{2.24}$$

The geometric properties of $F$ (theorem 2.6) imply $D_1 \leq D_0$ with equality if and only if $p = p^*$. As a consequence $d_{HD}(\mu(p))$ is maximal for $p = p^*$. ○

**Example 2.7 (Maximality of the Lyapunov Dimension of $\mu(p^*)$)** Massopust [Ma1, pages 7 and 27] proves, that $p^*$ maximizes the *Lyapunov dimension* $\Lambda(\mu(p))$ of the canonical dynamical system $\Phi$ associated with $\mu(p)$: $\Phi \,:\, K \times [0,1] \to K \times [0,1]$ where

$$\Phi(x,y) := (w_i(x), \frac{y - y_{i-1}}{p_i}) \quad \text{for } y \in [y_{i-1}, y_i]$$

with $y_0 := 0$ and $y_i = p_1 + \ldots + p_i$ (see also [GH1]). Massopust obtains

$$\Lambda(\mu(p^*)) = 1 + d_{\mathrm{box}}(K).$$

○

Finally note that

$$\mu(p^*) = c \cdot m^{D_0}|_K$$

for some normalization constant $c$. To see this, just check the invariance (2.10) of $m^{D_0}|_K$ using $m^{D_0}(w_i(K) \cap w_j(K)) = 0$ $(i \neq j)$ [Hut, page 737] and note that $m^{D_0}(K) \in ]0, \infty[$.
In the context of this subsection we would like to mention another special choice of probabilities $p_i$, which works for any IFS with affine maps $w_i$:

$$\overline{p}_j := \frac{\det(w_j)}{\sum_{i=1}^r \det(w_i)}.$$

The associated multifractal is sometimes called *geometrical multifractal* [Tél]. Growing structures with an underlying self-similar process [TV] may serve us as a first example: using only the number of particles in the various stages of the growth, one may determine 'mass-indices' $\alpha$ and corresponding 'fractal dimensions' $f(\alpha)$. This function equals exactly the multifractal spectrum $F$ of the underlying SMF endowed with 'geometrical' probabilities $\overline{p}_j$. The important aspect is the fact, that there is no measure needed to find $f$. Thus, this function characterizes the pure geometry of the system. As a second example let us consider the case of an IFS, for which the invariant set $K = \langle w_1, \ldots, w_r\rangle$ has positive $d$-dimensional Lebesgue measure (e.g. a triangle in the plane $\mathbb{R}^2$ [GH2, page 11]): choosing 'geometrical' probabilities $\overline{p}_j$ leaves one with $\mu$ equal to the normalized Lebesgue measure restricted to $K$.

### 2.4.2 Explicit Formulas

Most easily an explicit formula for $T(q)$ arises from (2.22) provided that all ratio numbers are equal, i.e. $\lambda_i = \lambda$ $(i = 1, \ldots, r)$. By simple calculation

$$T(q) = -\frac{1}{\log\lambda}\log\big(\sum_{i=1}^{r} {p_i}^q\big) \qquad \alpha(q) = -\frac{1}{\log\lambda}\frac{\sum_{i=1}^{r} {p_i}^q \log\big(\frac{1}{p_i}\big)}{\sum_{i=1}^{r} {p_i}^q}$$

$$F(\alpha(q)) = -\frac{1}{\log\lambda}\Big(\log\big(\sum_{i=1}^{r} {p_i}^q\big) + q\frac{-\sum_{i=1}^{r} {p_i}^q \log(p_i)}{\sum_{i=1}^{r} {p_i}^q}\Big).$$

So, the shape of the spectrum is independent of $\lambda$. Multifractals of this kind are widely used as simple examples ([HJKPS, HP, Falc4]) or as models of phenomena in nature [V1].

An explicit formula for $F(\alpha)$ can be extracted from the equation system (2.23), provided $r = 2$. Considering the variables $x_i = {p_i}^q {\lambda_i}^{\gamma - q\alpha}$ and setting $c_i = \log p_i - \alpha\log\lambda_i$ one finds

$$x_1 = \frac{c_2}{c_2 - c_1} \qquad \text{and} \qquad x_2 = \frac{c_1}{c_1 - c_2}.$$

By taking logarithms

$$F(\alpha) = \frac{c_2\log(-c_2) + (c_1 - c_2)\log(c_1 - c_2) - c_1\log(c_1)}{\log\lambda_1\log p_2 - \log\lambda_2\log p_1}$$

for $\alpha \in ]\alpha_\infty, \alpha_{-\infty}[$, where $\alpha_\infty = \log p_1/\log\lambda_1 < \log p_2/\log\lambda_2 = \alpha_{-\infty}$ without loss of generality.

Formulas free from the parameter $q$ have been presented until now only for special cases [EM, TV].

### 2.4.3 Atomic Multifractals

In this subsection degenerate probability- or ratio-numbers are considered, i.e. $p_i = 0$ or $\lambda_i = 0$.

**Example 2.8 (Vanishing Probabilities)** Let

$$K = \bigcap_{n=1}^{\infty}\bigcup_{\underline{i}\in I_n} V_{\underline{i}}$$

be an $r$-adic Cantor set (2.1) and take $t < r$ and $(p_1, \ldots, p_r)$ such that

$$\sum_{i=1}^{r} p_i = 1, \qquad p_i > 0 \quad (i = 1, \ldots, t) \qquad \text{and} \qquad p_i = 0 \quad (i = t+1, \ldots, r).$$



Then the product measure on $I_\infty = \{1, \ldots, r\}^{\mathbb{N}}$ associated with $(p_1, \ldots, p_r)$ has no longer all of $I_\infty$ as its support, but $I'_\infty = \{1, \ldots, t\}^{\mathbb{N}}$—considered as a subset of $I_\infty$. Thus

$$\operatorname{supp}(\mu) = K' = \bigcap_{n=1}^{\infty}\bigcup_{\underline{i}\in\{1,\ldots,t\}^n} V_{\underline{i}},$$

which is in general a proper subset of $K$. Moreover, $\mu = \pi'_* P'$, where $P'$ is the product measure on $I'_\infty$ associated with $(p_1, \ldots, p_t)$ and $\pi'$ is the restriction of $\pi$ to $I'_\infty$. Note that with the convention $0^0 := 0$ theorem 2.3 remains valid. ○

Thus, allowing degenerated probability vectors just means to extinguish certain sets in the construction (2.1) of $K$. In particular if $p_1 = 1$, then $\mu$ is the Dirac measure at the point $\pi(111\ldots)$ and hence atomic.

On the other hand, the following condition is sufficient to guarantee that a CMF is *nonatomic*, i.e. has no atoms:

**Lemma 2.7** *Assume that $\mu$ is an $r$-adic CMF, where vanishing probability numbers are expressively allowed here. Assume, on the other hand, that none of them equals one. Assume, furthermore, that the first two preconditions (2.15) and (2.16) of theorem 2.3 hold. Then $\mu$ is nonatomic.*

**Remark** As a consequence, SMFs with $r \geq 2$ are nonatomic.

**Proof** It is well known that the atoms of Radon measures must be singletons. As we will see, the preconditions of this lemma imply that singletons are nullsets. Thus the proof is complete. However, we give the full argument.

**i)** Assume there is an atom $A$, i.e. $A$ is measurable, $0 < \mu(A) < \infty$ and for any measurable subset $E$ of $A$ either $\mu(E) = 0$ or $\mu(E) = \mu(A)$. An application of Zorn's lemma to the set $\{E \subset A : E \text{ measurable and } \mu(E) = \mu(A)\}$ endowed with the inclusion as an ordering yields a minimal measurable subset $B$ of $A$ with the same measure as $A$. By its minimality and the properties of $A$ every measurable subset $E$ of $B$ must be a null set.

**ii)** Every singleton $\{x\}$ is a Borel set and hence measurable. Moreover, it is a $\mu$-nullset. To see this note first that $\pi^{-1}(x)$ is a finite set: Take $b_1$ from (2.21) and assume that $\pi^{-1}(x)$ contains more than $b_1$ infinite words, say $\underline{i}_\infty^{(k)}$ $(k = 0, \ldots, b_1)$. For sufficiently large $m$ the initial segments $(\underline{i}_\infty^{(k)}|m)$ are distinct. Now take $\delta$ small enough to guarantee $|\underline{j}| \geq m$ for all $\underline{j} \in J_\delta$. Since $J_\delta$ is secure and tight, there is for every $k = 0, \ldots, b_1$ a (unique) number $n_k$ s.t. $\underline{j}^{(k)} := (\underline{i}_\infty^{(k)}|n_k)$ lies in $J_\delta$. But the corresponding $b_1 + 1$ sets $V_{\underline{j}^{(k)}}$ contain $x$, i.e. intersect the $\delta$-box containing $x$, in contradiction to (2.21). Thus $\pi^{-1}(x)$ is indeed finite and hence a $P$-nullset due to $p_i < 1$ for all $i$.

**iii)** Finally, the fact that every singleton $\{x\}$ is a null set contradicts the minimality of $B$: taking away a point from $B$ yields a smaller, measurable set with the same measure. Thus there is no atom. ◇

Less trivial is the case where one allows vanishing ratio numbers. Generalizations of the short example below are immediate.

**Example 2.9 (Vanishing Ratios)** Consider the IFS $w_1(x) = x/2$, $w_2(x) \equiv 1$ and an arbitrary probability vector $(p_1, p_2)$. The support of the resulting CMF $\mu$ is

$$K = \{0\} \cup \{2^{-n} : n \in \mathbb{N}_0\},$$

and $\mu$ itself is a linear combination of Dirac measures concentrated at the points $2^{-n}$:

$$\mu = \sum_{n=0}^{\infty} p_2 p_1^n \cdot \delta_{\{2^{-n}\}}.$$

So, this is another atomic measure arising from an IFS.
To calculate $T(q)$ it is convenient to use the admissible sequence $\delta_n = 2^{-n}$. By direct computation of the measures $\mu((B)_1)$

$$\begin{aligned} S_{\delta_n}(q) &= {p_1}^{nq} + {p_1}^{(n-1)q} + ({p_1}^n p_2 + {p_1}^{n-1} p_2)^q + \sum_{k=0}^{n-2} ({p_1}^k p_2)^q \\ &= c \cdot {p_1}^{nq} + {p_2}^q (1 - {p_1}^q)^{-1} \end{aligned}$$

with some $c \geq 1$, and

$$T(q) = \begin{cases} q \cdot \log p_1 / \log 2 & \text{if } q < 0, \\ 0 & \text{otherwise.} \end{cases}$$

In this example the formula (2.22) holds exactly for $q \leq 0$. Thus, $\lambda_i \neq 0$ is a necessary precondition in corollary 2.5, since $O = ]0, 2[$ is a basic open set for the above IFS. Note that theorem 2.3 does not apply since (2.15) and (2.17) cannot be satisfied. But still $K$ may be constructed starting with $V_{\text{nil}} = [0, 1]$ and using the IFS as usual. Thus $\mu$ is contractive with $\lambda_1 = 1/2$ and $\lambda_2 \in ]0, 1[$ arbitrary, and corollaries 2.1 and 2.2 apply. Indeed $T(0) = 0$ is the box dimension of $K$, $T$ is continuous and $D_q$ is bounded by $-\log p_1 / \log 2$.
For later use we note that $F$ equals the Legendre transform of $T$, i.e.

$$F(\alpha) = F^m(\alpha) = \begin{cases} 0 & \text{if } 0 \leq \alpha \leq -\log p_1 / \log 2, \\ -\infty & \text{otherwise.} \end{cases}$$

Proof: due to proposition 1.15 $F^+(\alpha)$ and $F^-(\alpha)$ only take the values $-\infty$ and 0 with the discontinuities at $D_\infty = 0$ resp. at $D_{-\infty} = -\log p_1 / \log 2$. Since $F \leq F^m$, it remains only to show that $F(\alpha) \geq 0$ for $\alpha \in ]D_\infty, D_{-\infty}[$. Fix such an $\alpha$ and take an arbitrary $\kappa > 0$. Choose the sequence $\delta_n = (p_2 {p_1}^n)^(1/\alpha)$ and take $n$ large enough to ensure $\delta_n < (3(1+\kappa))^{-1} 2^{-n}$ (note that $p_1 < 2^{-\alpha}$). Now there is a unique box $B_n$ in $G_{\delta_n}$ which contains the point $2^{-n}$. By the choice of $n$ the enlarged interval $(B_n)_\kappa$ does not contain any other point of the form $2^{-m}$. Thus $\mu((B_n)_\kappa) = p_2 {p_1}^n = {\delta_n}^\alpha$, which proves the claim. ○



### 2.4.4 Subsets of Given Local Hölder Exponent

The definition of the spectrum $F$ gives credit to the intuition that $F(\alpha)$ is the dimension of a certain subset $K_\alpha$ of $K$. This set is roughly described by the property that the measure of a ball with center in $K_\alpha$ and diameter $\delta$ scales as $\delta^\alpha$ for $\delta \downarrow 0$. So far we are not aware of either a general proof justifying this intuition nor of a counterexample. But since this view of things helps to understand what kind of information about $K$ is provided by the spectrum $F$, we feel obliged to report on a few cases where the above interpretation is valid.
A first, almost trivial example is the following subset of the support $K$ of a SMF $\mu$:

$$C_\infty := \langle w_1, \ldots, w_t \rangle,$$

where the similarities $w_i$ are ordered in a way to assure $\log p_1 / \log \lambda_1 = \ldots = \log p_t / \log \lambda_t < \log p_i / \log \lambda_i$ for all $i \geq t+1$. $C_\infty$ is the set of the 'most probable' points (see [Tél, HJKPS]). It is self-similar and has by (2.13) and theorem 2.6 the dimension

$$d_{\text{box}}(C_\infty) = d_{HD}(C_\infty) = F(D_\infty).$$

Note that this interpretation of $F(D_\infty)$ is less immediate for self-affine multifractals (see Ex. 3.3). The two trivial cases are $t = 1$ ($C_\infty$ is a singleton) and $t = r$ ($C_\infty = K$, $\mu$ is homogeneous). In a similar fashion, the set of the 'most rarefied' points $C_{-\infty}$ can be defined. Provided (2.22) holds for negative $q$, the dimension of $C_{-\infty}$ is $F(D_{-\infty})$.

**Example 2.10 (Subsets of Local Hölder Exponent $\alpha$. Disjoint Case.)** Reporting shortly on [CM] we define for a given SMF $\mu = \langle\langle \lambda_1, \ldots, \lambda_r; p_1, \ldots, p_r \rangle\rangle$

$$\hat{K}_\alpha := \{\underline{i}_\infty \in I_\infty : \lim_{n\to\infty} \frac{\log p_{(\underline{i}_\infty | n)}}{\log \lambda_{(\underline{i}_\infty | n)}} = \alpha\} \qquad K_\alpha := \pi(\hat{K}_\alpha).$$

Defining $\beta(q)$ as the unique solution $\gamma$ of (2.14) for each $q$, the authors of [CM] are able to prove that

$$d_{HD}(K_{\alpha(q)}) = \beta(q) + q\alpha(q)$$

for all $q$, where $\alpha(q) = -\beta'(q)$. This result is a convincing example for the power of an approach 'tailored to multiplicative cascades' and the use of symbolic dynamics. For a geometric relevance, however, the following characterization of $K_\alpha$ in terms of the local behaviour of the measure $\mu$ is essential: *Provided the sets $w_i(K)$ are pairwise disjoint*

$$K_\alpha = \{x \in K : \lim_{\delta \downarrow 0} \frac{\log \mu(U(x, \delta))}{\log \delta} = \alpha\}.$$

(This is an almost immediate consequence of the fact that the $\varepsilon$-neighbourhood of $K$ satisfies the OSC for sufficiently small $\varepsilon$. However, a proof is also contained in [CM].) Consequently $K_\alpha$ is just the set of all points with *local Hölder exponent $\alpha$* [EM]. With corollary 2.5 this can be summarized as follows:

**Proposition 2.8 (Cawley-Mauldin)** *Let $\mu$ be a SMF. Provided the sets $w_i(K)$ are mutually disjoint,*

$$d_{HD}(\{x \in K : \lim_{\delta \downarrow 0} \frac{\log \mu(U(x,\delta))}{\log \delta} = \alpha\}) = F(\alpha)$$

*for all $\alpha \in ]D_\infty, D_{-\infty}[$. Moreover, $\mu(K_1) = 1$ ([CM, p 210, remark 2.12]).*

○

The next example shows, that disjointness is not necessary to achieve similar results.

**Example 2.11 (Subsets of Local Hölder Exponent $\alpha$. One Dimensional Case.)** A result of Collet et al [CLP] will be adopted to fit our purpose. Take a self-similar set $K = \langle w_1, \dots, w_r \rangle$ on $\mathbb{R}$ with the following two properties:
a) an interval $O$ satisfies the OSC, and
b) the map

$$g : x \mapsto w_i^{-1}(x) \qquad \text{for } x \in w_i(\overline{O}) = V_i$$

is continuous on $K_1 = \cup_{i=1,\dots,r} w_i(\overline{O})$.
As an example take $w_1(x) = x/3$, $w_2(x) = -x/3 + 2/3$ and $w_2(x) = x/3 + 2/3$ with $K = [0, 1]$. Now supplement $K$ to a SMF $\mu = \langle w_1, \dots, w_r; p_1, \dots, p_r \rangle$ with arbitrary probability vector. It is easily verified that $\mu$ satisfies the hypothesis of [CLP].
Trying to avoid the difficulties arising in the calculation of the singularity exponents $\tau(q)$ for negative $q$, the authors of [CLP] introduce certain partition sums $Z_n(q)$ related to the $2^{-n}$-grid. A closer look at the sophisticated construction reveals inequalities between $Z_n(q)$ and $S_\delta(q)$, which are of the same kind as used in the proof of proposition 1.5. Consequently, $T(q) = \lim_{n\to\infty} \log Z_n(q)/\log 2^n$. So, [CLP, theorem 3.1] and our corollary 2.4 yield:

**Proposition 2.9 (Collet et al.)** *Let $\mu$ be as above. Then*

$$d_{HD}\Big(\{x \in K : \lim_{\substack{|E|\to 0 \\ x\in \mathrm{int}(E)}} \frac{\log \mu(E)}{\log |E|} = \alpha\}\Big) = F(\alpha)$$

*for $\alpha \neq \alpha_0$ from $]D_\infty, D_{-\infty}[$. Thereby $|E|$ denotes the length of the interval $E$.*

○

For reasons of completeness we point to further results in this field obtained by Lopes [Lop] for Julia sets of hyperbolic rational maps in the plane, by Schmeling et Siegmund-Schultze [S] for self-affine measures and by Rand [R] for certain CMF.
There is more to say about the measures $\mu$ occurring in examples 2.10 and 2.11. Since SMFs are nonatomic (lemma 2.7), a famous theorem of Young [Y, p. 112] allows the following reasoning: taking any $\mu$ as in example 2.10 and any $\alpha$ with



$\mu(K_\alpha) > 0$, [Y] implies $d_{HD}(K_\alpha) = \alpha$ and hence $F(\alpha) = \alpha = \alpha_1 = D_1$. On the other hand, $d_{HD}(K_\alpha) = F(\alpha) < D_0$ for all $\alpha \neq \alpha_0$. Thus

$$\mu(K_\alpha) = 0 \quad (\alpha \neq \alpha_1) \qquad m^{D_0}(K_\alpha) = 0 \quad (\alpha \neq \alpha_0), \tag{2.25}$$

The same holds for $\mu$ as in example 2.11.
From (2.25) the difference between the invariant measure and the $D_0$-dimensional Hausdorff measure (restricted to $K$ and normalized) becomes apparent. Moreover, considering these measures as probability measures on the embedding space, the local Hölder exponent takes the value $\alpha_1$, respectively $\alpha_0$, with probability one.
The first equation in (2.25) supports the imagination that the measure $\mu$ concentrates in the $\delta$-boxes with $\mu(B) \simeq \delta^{D_1}$. Compare also page 28, (2.24) and proposition 2.8.

**Example 2.12 (Concentration of $\mu$)** Assume that $\mu$ is a multifractal and $\bar\alpha \geq 0$ such that for any $h > 0$ there is a $\eta > 0$ with $F^+(\alpha) \leq \alpha - \eta$ for all $\alpha \notin [\bar\alpha - h, \bar\alpha + h]$. Take e.g. a SMF $\mu$ and choose $\bar\alpha = D_1$.
Then the calculation of Falconer [Falc4, p. 260] is valid, since $f(\alpha) \leq f^+(\alpha) = F^+(\alpha)$. It shows that for $G_\delta(h) := \{B \in G_\delta : \delta^{\bar\alpha + h} \leq \mu(B) \leq \delta^{\bar\alpha - h}\}$,

$$\mu(\bigcup_{B\in G_\delta(h)} B) \to 1 \quad (\delta \downarrow 0) \qquad \text{for any } h > 0.$$

Thus $\mu$ is concentrated in the boxes with $\mu(B) \simeq \delta^{\bar\alpha}$. But note that these boxes do not necessarily form a decreasing sequence of compact sets as $\delta \downarrow 0$. ○

Finally we give the example of a SMF $\mu$ and a subset $E$ of its support $K$ with $d_{HD}(E) = D_1$ and draw the connection to (2.24) and to the examples 2.10 and 2.12.

**Example 2.13 (Subset of Exact Dimension $D_1$)** The following result is due to Eggleston [Eggl]:
*Let $(p_o, \dots, p_{r-1})$ be a probability vector and denote by $N(x, n, k)$ the number of times, the digit $k$ occurs amongst the first $n$ digits of the $r$-adic decimal expansion of $x \in [0, 1]$. Then the Hausdorff dimension of*

$$E = \{x \in [0, 1] : \lim_{n\to\infty} \frac{1}{n} N(x, n, i) = p_i,\ i = 0, \dots, r-1\}$$

*satisfies*

$$d_{HD}(E) = \frac{\sum_{i=0}^{r-1} p_i \log p_i}{-\log r}.$$

(Note that only countable many points $x$ have more than one decimal representation. Concerning matters of dimension greater than zero, enumerable sets are negligible.)

Eggleston's result may be regarded from the point of view of multifractal formalism: Choose $\mu = \langle w_o, \ldots, w_{r-1}; p_o, \ldots, p_{r-1}\rangle$ with $w_i(x) = (i+x)/r$. Of course $K = [0,1]$ and $E$ is a subset of $K$. Since countable sets are $P$-null sets,

$$\pi(\underline{i}_\infty) \in E \quad \Leftrightarrow \quad \forall k = 0, \ldots, r-1 \quad \lim_{n\to\infty} \frac{1}{n} \#\{l \le n \,:\, i_l = k\} = p_k,$$

for $P$-almost every infinite word $\underline{i}_\infty$. By the Strong Law of Large Numbers the last condition holds $P$-almost sure. Thus $\mu(E) = 1$. By corollary 2.4 and by Eggleston's result $d_{HD}(E) = D_1$, which by (2.24) may be expected to equal $d_{HD}(\mu)$.
Finally, $E$ can be related to the examples above. For $\mu$-almost every $x \in E$ there is exactly one $\underline{i}_\infty \in \pi^{-1}(\{x\})$. Then, letting $\delta_n = r^{-n}$, there is a unique $B_n \in G_{\delta_n}$ which contains $x$. Moreover, the closure of $B_n$ is $V_{(\underline{i}_\infty|n)}$ and

$$\mu(B_n) = p_{(\underline{i}_\infty|n)} = \prod_{i=0}^{r-1} p_i{}^{N(x,n,i)},$$

leading to

$$\frac{\log \mu(B_n)}{\log \delta_n} = \frac{\log p_{(\underline{i}_\infty|n)}}{\log \lambda_{(\underline{i}_\infty|n)}} = \frac{\frac{1}{n}\sum_{i=0}^{r-1} N(x,n,i)\log p_i}{-\log r} \to \frac{\sum_{i=0}^{r-1} p_i \log p_i}{-\log r} = D_1$$

as $n \to \infty$. The connection to example 2.12 is immediate. Furthermore, $E = K_{D_1}$ up to a $\mu$-nullset. But there is no rigorous argument showing that $E$ is the set of local Hölder exponent $D_1$. ○

See also example 3.3 for a self-affine multifractal and an explicit subset $K'$ of 'strict local Hölder exponent' $D_1$.

### 2.4.5 Counterexamples

This subsection provides four multifractals with unusual spectrum:

- A so-called left-sided spectrum [MEH], which means that $T(q) = \infty$ for all negative $q$. Consequently $\alpha^* = \infty$ and here even $T(0) \neq \sup F^+(\alpha)$. For further examples of this kind see [MEH, ME, CJVP].
- Nonconcave spectra.
- Non grid-regular singularity exponents and spectrum.

**Example 2.14 (A Multifractal $\mu$ with $F^-(\alpha) \equiv d_{\text{box}}(K)$.)**
In contrast with example 2.1 the extraordinary behaviour of $F^-$ does not arise from an inherent inability of the multifractal formalism but reflects the strong inhomogeneity of the measure $\mu$ and the existence of arbitrarily large local Hölder exponents.



Let $K$ be the middle third Cantor set as constructed in example 1.1. Unlike the construction of CMFs the product measure $P'$ on its codespace $\{1,2\}^{\mathbb{N}}$ is chosen according to the more and more one-sided measures

$$\{1\} \mapsto p_1^{(n)} = 3^{-n^2} \qquad \{2\} \mapsto p_2^{(n)} = 1 - 3^{-n^2}$$

on the $n$-th factor $\{1,2\}$. This is not a multiplicative cascade any more.
Let $\mu := \pi_* P'$. The calculation of $F^-$ is carried out using the admissible sequence $\delta_n = 3^{-n}$. Singletons are $\mu$-nullsets. So for any $\delta_n$-box $B$

$$\mu(B) = \begin{cases} p_{i_1}^{(1)} \cdot \ldots \cdot p_{i_n}^{(n)} = \mu((B)_1) & \text{if } \overline{B} = V_{\underline{i}} \text{ and } \underline{i} \in I_n \\ 0 & \text{otherwise.} \end{cases}$$

Note first that $p_{i_1}^{(1)} \cdot \ldots \cdot p_{i_n}^{(n)} \le p_1^{(n)} = \delta_n{}^n$ for all $\underline{i} \in I_n$ with $i_n = 1$. Now fix $\alpha$. For any integer $n > \alpha$

$$2^n = \# I_n \ge M_{\delta_n}(\alpha) \ge \#\{\underline{i} \in I_n \,:\, i_n = 1\} = 2^{n-1}.$$

Together with (2.13):

$$F^-(\alpha) = \lim_{n\to\infty} \frac{\log M_{\delta_n}(\alpha)}{-\log \delta_n} = \frac{\log 2}{\log 3} = d_{\text{box}}(K),$$

as it was claimed. In addition

$$S_{\delta_n}(q) = \prod_{k=1}^{n} \left(3^{-k^2 q} + (1 - 3^{-k^2})^q\right) = c(n_1) \prod_{k=n_1}^{n} \left(3^{-k^2 q} + (1 - 3^{-k^2})^q\right).$$

For $q > 0$ and $\varepsilon > 0$ choose $n_1$ large enough to ensure

$$3^{-k^2 q} \le \varepsilon (1 - 3^{-k^2})^q \quad \text{and} \quad -2 \le 3^{k^2} \log(1 - 3^{-k^2}) \le -\frac{1}{2} \quad \text{for } k \ge n_1.$$

Straightforward estimates give

$$\frac{-2q}{3^{n_1^2}}(n - n_1 + 1) \le \log S_{\delta_n}(q) - \log c(n_1) \le (n - n_1 + 1)\big(\log(1+\varepsilon) - \frac{q}{2 \cdot 3^{n^2}}\big).$$

Dividing first by $-\log \delta_n$, and letting then $n \to \infty$, then $n_1 \to \infty$ and finally $\varepsilon \to 0$ shows that $T(q)$ vanishes and is grid-regular. For $q = 0$ direct computation yields $S_{\delta_n}(0) = 2^n$ and $T(0) = d_{\text{box}}(K)$. For $q < 0$, proposition 1.15 and lemma 1.16 yield the grid-regular value $T(q) = \infty$.
Finally, $F^+(\alpha) \le 0$ for all positive $\alpha$ due to $F^+(\alpha) \le T(q) + q\alpha$. On the other hand theorem 1.1 and (1.24) give $F(0) = F^+(0+) = 0$. (A straightforward calculation yields even $F(\alpha) \ge 0$ for all $\alpha > 0$.) By monotonicity $F^+(\alpha) = 0$ for all strictly positive $\alpha$. So it has the looks of the positive semispectrum of a Dirac measure. This reflects the fact that the measure $\mu$ is concentrated at the point 1. Only $F^-$ and $T$ reveal some more information about $\mu$.
Summarizing, this multifractal exhibits the following features:

- The singularity exponents are not even semicontinuous at zero.

$$T(q) = \begin{cases} \infty & \text{if } q < 0 \\ \log 2/\log 3 & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

- The multifractal is left-sided, hence $\alpha^* = \infty$. Moreover,

$$F(\alpha) = F^+(\alpha) = 0 \quad (\alpha > 0) \qquad \text{and} \qquad F^-(\alpha) = d_{\mathrm{box}}(K) \quad (\alpha \in \mathbb{R}).$$

- $T(0) > \sup F(\alpha) = \sup F^m(\alpha) = 0.$

○

**Lemma 2.10** *Let $\mu_1, \dots, \mu_t$ be multifractals with mutually disjoint supports and let $(c_1, \dots, c_t)$ be a probability vector. Then*

$$\mu := \sum_{i=1}^{t} c_i \mu_i$$

*is again a multifractal. Moreover, using a selfexplanatory notation,*

$$F(\alpha) = \max_{i=1,\dots,t} (F_i(\alpha)) \qquad F^{\pm}(\alpha\pm) = \max_{i=1,\dots,t} (F_i^{\pm}(\alpha\pm))$$

$$T(q) = \max_{i=1,\dots,t} (T_i(q)) \qquad \underline{T}(q) = \max_{i=1,\dots,t} (\underline{T}_i(q)).$$

*and grid-regularity of the maximal $F_i(\alpha)$ resp. $T_i(q)$ carries over to $F(\alpha)$ resp. $T(q)$.*

**Remark** Example 2.9 shows that the conclusions are wrong for infinite sums of multifractals. (All Dirac measures possess the same trivial function $T(q) \equiv 0$. An infinite linear combination of them, however, may not.)

**Proof** Throughout the proof it will be assumed that $0 < 3\sqrt{d}\cdot\delta < \mathrm{dist}(K_i, K_j)$ for all distinct $i$ and $j$. Thus, if $B \in G_\delta$, there is a unique $i$ such that $\mu((B)_1) = c_i \mu_i((B)_1)$. First the proof is given for $F(\alpha)$. Fix $\alpha$ and take $\varepsilon > 0$ arbitrarily. Choose $\delta_0 > 0$ such that ${\delta_0}^{\varepsilon} \le c_i \le {\delta_0}^{-\varepsilon}$ for $i = 1, \dots, t$. With a selfexplanatory notation it follows that

$$N_\delta(\alpha+\varepsilon) - N_\delta(\alpha-\varepsilon) \le \sum_{i=1}^{t} N_\delta^{(i)}(\alpha+2\varepsilon) - N_\delta^{(i)}(\alpha-2\varepsilon) \le N_\delta(\alpha+3\varepsilon) - N_\delta(\alpha-3\varepsilon)$$

whenever $0 < \delta < \delta_0$. This implies immediately the claim. The assertion for the semispectra follows similarly from

$$N_\delta(\alpha+\varepsilon) \le \sum_{i=1}^{t} N_\delta^{(i)}(\alpha+2\varepsilon) \le N_\delta(\alpha+3\varepsilon).$$

To treat $T(q)$ and $\underline{T}(q)$ note that $\min(c_i^q) \cdot S_\delta^{(k)}(q) \le S_\delta(q) \le \max(c_i^q) \cdot \sum S_\delta^{(i)}(q)$. ◇



**Example 2.15 (Nonconcave Spectrum)** Take

$$w_1(x) = \frac{x}{3} \quad w_2(x) = \frac{2+x}{3} \qquad t_1(x) = \frac{4+x}{3} \quad t_2(x) = \frac{6+x}{3}$$

and set $\mu_1 = \langle w_1, w_2; 2/3, 1/3\rangle$, $\mu_2 = \langle t_1, t_2; 8/9, 1/9\rangle$ and $\mu := 1/2(\mu_1 + \mu_2)$. Since $\mu_1$ and $\mu_2$ are symmetric biadic SMFs (i.e. $r = 2$ and $\mathrm{Lip}(w_1)$=$\mathrm{Lip}(w_2)$), their spectra are symmetric with respect to the corresponding extrema, say ${\alpha_0}'$ and ${\alpha_0}''$. Moreover, their extremal values are both equal to $\log 2/\log 3 = d_{\mathrm{box}}(K_1) = d_{\mathrm{box}}(K_2)$. By lemma 2.10 $\mu$ has the spectrum shown in figure 2.5. Note that $F$ is grid-regular and that it equals $F^m$ exactly for $\alpha$ outside $]{\alpha_0}', {\alpha_0}''[$. ○

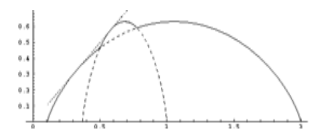

Figure 2.5: *The nonconcave spectrum of $\mu = 1/2(\mu_1 + \mu_2)$ as given in example 2.15. The dashed parts show the internal bisector of the axes and the spectra of $\mu_1$ and $\mu_2$.*

**Example 2.16 (Another Nonconcave Spectrum)** Take

$$w_1(x) = \frac{x}{3} \quad w_2(x) = \frac{2+x}{3} \qquad t_1(x) = \frac{16+x}{9} \quad t_2(x) = \frac{24+x}{9}$$

and set $\mu_1 = \langle w_1, w_2; 2/3, 1/3\rangle$, $\mu_2 = \langle t_1, t_2; 5/9, 4/9\rangle$ and $\mu := 1/2(\mu_1 + \mu_2)$. By lemma 2.10 the spectrum of $\mu$ is as shown in figure 2.6. Moreover, it is grid-regular and equals $F^m$ in the union of two disjoint intervals. ○

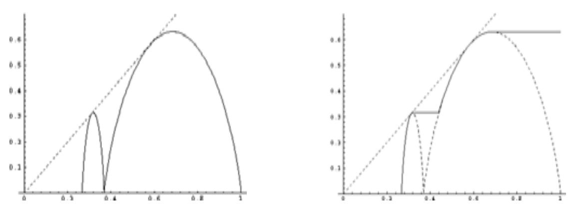

Figure 2.6: *The nonconcave spectrum (on the left) of another multifractal $\mu = 1/2(\mu_1 + \mu_2)$ and its positive semispectrum $F^+$. See Ex. 2.16. The dashed parts show the internal bisector of the axes and the spectra of $\mu_1$ and $\mu_2$.*

The last two examples allow several conclusions.

- First of all, a spectrum $F$ is not necessarily concave. Moreover, it is not necessarily differentiable.
- Secondly, neither $F$ nor $F^m$ must everywhere equal the Legendre transform of $T$. This violates the duality between $F$ and $T$: the latter equals always the transform of $F^m$ (except maybe at 0) and is hence convex. This asymmetry is also reflected in the fact that taking the maximum of functions conserves convexity but not concavity. To say it in a different way: $T$ is more regular than $F$ since it depends on the values $\mu((B)_1)$ through a sum or 'average'. On the other hand, $T$ provides a coarser way of measuring the singularities of $\mu$ and carries less information than $F$.
- As a third point consider lemma 1.10: $F$ may well be strictly monotonous in an open interval and nevertheless equal neither $F^+$ nor $F^-$. Thus a 'dual' version of the mentioned lemma does not hold. Furthermore, $F^+(\alpha) < F^-(\alpha)$ is not enough to imply $F(\alpha) = F^+(\alpha)$.
- Finally, the touching point of $F$ with the internal bisector of the axes is not necessarily unique, and $T$ is not necessarily differentiable at $q = 1$. Consequently, the $\lim_{q\to 1} D_q$ may not exist.

**Example 2.17 (Non Grid-Regular Exponents)** A biadic CMF on $\mathbb{R}$ with $\underline{T}(q) < T(q)$ $(q \neq 1)$ will be constructed. Choose two strictly increasing sequences of integers $(n_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ such that

$$\frac{n_k}{n_k + m_{k-1}} \to 1 \qquad \frac{n_k}{n_k + m_k} \to 0 \quad (k \to \infty).$$

Let $l(n)$ denote the number of members of $(n_k)_{k\in\mathbb{N}}$ less or equal to $n$. The sets $V_{\underline{i}}$ involved in the construction are of the form

$$V_{\underline{i}} = [a_{\underline{i}} 3^{-M}, (a_{\underline{i}} + 1)3^{-M}]$$

for some integer $a_{\underline{i}}$, where $M = M(n) := n + m_{l(n)}$ and $n = |\underline{i}|$. To be more precise, set $V_{\text{nil}} = [0,1]$ and $m_0 := 0$. Assuming that $V_{\underline{i}}$ is constructed and of the form as above, let $a_{\underline{i}*1}$ and $a_{\underline{i}*2}$ by defined through

$$\begin{aligned} a_{\underline{i}*1} \cdot 3^{-M(n+1)} &= a_{\underline{i}} 3^{-M(n)} \\ a_{\underline{i}*2} \cdot 3^{-M(n+1)} &= a_{\underline{i}} 3^{-M(n)} + 2 \cdot 3^{-M(n)-1}. \end{aligned}$$

Since the function $n \mapsto M(n)$ increases by $m_k - m_{k-1} + 1$ when $n = n_k$, and by 1 otherwise, this choice of $V_{\underline{i}}$ actually means the following: the well known construction of the middle third Cantor set is carried out until a stagenumber $n = n_k$ is reached. Then the left boundary points $a_{\underline{i}}$ of the $V_{\underline{i}}$ are chosen as usual, but the length extremely small relative to the length of the preceding intervals. It is immediate that the $V_{\underline{i}*k}$ are subsets of $V_{\underline{i}}$.



Thus the construction of $K$ is complete. Let $\mu = \langle \pi; p_1, p_2 \rangle$ where $\pi$ is the coordinate map of $K$ and $(p_1, p_2)$ an arbitrary probability vector. For the sake of definiteness assume $p_1 \geq p_2$.
Take an arbitrary integer $t$ and set $\delta_t = 3^{-t}$. There is a unique integer $N = N(t)$ defined by

$$M(N-1) + 1 \leq t \leq M(N).$$

Every set $V_{\underline{j}}$ with $|\underline{j}| = N - 1$ belongs to the $3^{-M(N-1)}$-grid and its middle third interval with length $3^{-M(N-1)-1}$ separates its two 'daughter intervals' $V_{\underline{j}*1}$ and $V_{\underline{j}*2}$. By induction each set $V_{\underline{i}}$ with $|\underline{i}| = N$ is contained in a $3^{-t}$-box, just because $3^{-M(N)} \leq 3^{-t}$; furthermore, since $3^{-t} \leq 3^{-M(N-1)-1}$, every two distinct $V_{\underline{i}}$ with $|\underline{i}| = N$ are separated by a $3^{-t}$-box which does not meet $K$. Consequently, for any $\delta_t$-box $B$ either $\mu(B) = 0$ or $\mu((B)_1) = \mu(B) = \mu(V_{\underline{i}}) = p_{\underline{i}}$ for the unique $\underline{i} \in I_N$ for which $V_{\underline{i}}$ is contained in $\overline{B}$. This yields

$$\frac{\log S_{\delta_t}(q)}{-\log \delta_t} = \frac{\log\Big(\sum_{|\underline{i}|=N(t)} p_{\underline{i}}{}^q\Big)}{t \log 3} = \frac{N(t)}{t} \frac{\log(p_1{}^q + p_2{}^q)}{\log 3}.$$

Taking first $t_k = n_k + m_{k-1}$ results in $N(t_k) = n_k$ and $N(t_k)/t_k \to 1$ $(k \to \infty)$. Taking $t'_k = n_k + m_k$ implies again $N(t'_k) = n_k$, but $N(t'_k)/t'_k \to 0$ $(k \to \infty)$. Since $0 \leq N(t)/t \leq 1$ for all integers $t$, this results in

$$T(q) = \max\Big(\frac{\log(p_1{}^q + p_2{}^q)}{\log 3}, 0\Big) \quad \underline{T}(q) = \min\Big(\frac{\log(p_1{}^q + p_2{}^q)}{\log 3}, 0\Big) \qquad (q \in \mathbb{R}).$$

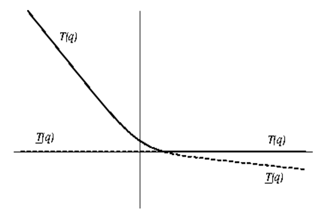


Figure 2.7: *A CMF with* $\underline{T}(q) < T(q)$ $(q \neq 1)$ *(Ex. 2.17 with* $p_1 = .85$, $p_2 = .15$*).*

Finally, applying the arguments given in the proof of theorem 1.2 to the sequences $(\delta_{t_k})_{k\in\mathbb{N}}$ and $(\delta_{t_k'})_{k\in\mathbb{N}}$ shows that

$$\begin{aligned} F^-(\alpha) > \liminf_{\delta\downarrow 0} \frac{\log M_\delta(\alpha)}{-\log \delta} &= -\infty \quad \text{for } 0 < \alpha < -\log p_2/\log 3 \\ F^+(\alpha) > \liminf_{\delta\downarrow 0} \frac{\log N_\delta(\alpha)}{-\log \delta} &= \begin{cases} 0 & \text{for } \alpha > -\log p_1/\log 3 \\ -\infty & \text{for } 0 < \alpha < -\log p_1/\log 3. \end{cases} \end{aligned}$$

○

# Chapter 3

# Self-Affine Multifractals

Having treated the self-similar case, one way to go further is to consider affinities. In fact an intensive study of self-affine sets has led to important results [Falc3, Falc5, K, U, Z, GL] and applications, the latter mostly in the field of fractal interpolation [BEHM, GH2, Ma2, Bed4]. It is, therefore, natural to consider self-affine measures. In doing so we will restrict our investigation to a particular kind of affinities and denote the obtained self-affine multifractals by SAMF. It is almost evident that the characteristic values of the involved affinities will partly determine the singularity exponents $T(q)$, like in the self-similar case. Carrying the analogy even further, one may expect implicit equations to hold for $T(q)$, from which the box dimension is recovered for $q = 0$. This indeed turns out to be the case. However, a completely new feature appears: the singularity exponents $T(q)$ of a SAMF are obtained as the maximum of the solutions of two equations. This is a consequence of the fact that the affine maps under consideration stretch with different ratios in two fixed invariant subspaces.

In the first section the definition of SAMFs is given and it is shown how the asymptotic behaviour of $S_\delta(q)$—and thus $T(q)$—is governed by the probability numbers and the characteristic values of the involved maps. Section two is devoted to limit theorems needed in section three, where the singularity exponents $T(q)$ are computed and grid-regularity and differentiability are discussed. Falconer gave the 'almost sure' dimension of self-affine sets. Section four provides this value in the context of this chapter and compares it with the actual box dimension $D_0 = T(0)$. In section five examples are developed and relations are drawn to recent publications.

## 3.1 Geometric Properties

In the first half of this section the particular geometrical situation of the self-affine measures under consideration is introduced. Then, after drawing first consequences, an intuitive understanding of the asymptotic behaviour of $S_\delta(q)$ is provided on

page 72. In the second half of the section this intuitive argument is made rigorous step by step.
For $i = 1, \ldots, r$ let $w_i$ be a *diagonal affine contraction* of $\mathbb{R}^2$, i.e.

$$w_i : (x^{(1)}, x^{(2)}) \mapsto (\vartheta_i \lambda_i x^{(1)} + u_i, \zeta_i \nu_i x^{(2)} + v_i) \tag{3.1}$$

where $\vartheta_i$ and $\zeta_i$ are from $\{-1, +1\}$, $u_i$ and $v_i$ from $\mathbb{R}$, and where

$$\begin{aligned} \lambda :=& \ \max\{\lambda_1, \ldots, \lambda_r, \nu_1, \ldots, \nu_r\} \ < 1, \\ \nu :=& \ \min\{\lambda_1, \ldots, \lambda_r, \nu_1, \ldots, \nu_r\} \ > 0. \end{aligned} \tag{3.2}$$

Similarly as for SMFs it is required to have a nonempty, bounded, connected open set $O$ such that

$$w_i(O) \subset O \quad (i = 1, \ldots, r) \qquad \text{and} \qquad w_i(O) \cap w_j(O) = \emptyset \quad (i \neq j). \tag{3.3}$$

In order to treat affinities rather than similarities one more regularity condition is needed: denote by $R$ the smallest closed rectangle with sides parallel to the axes, which contains $O$. For the sake of simplicity $R = [0,1]^2$ will be assumed. This choice is not really a restriction as far as multifractal formalism is concerned: any rectangle can be transformed to $[0,1]^2$ by a diagonal affine map $\Phi$. Moreover, $\Phi(K)$ is invariant under $\Phi \circ w_i \circ \Phi^{-1}$, which possess the same characteristical values as $w_i$. The additional hypothesis on $O$ is: there is a $\varrho > 0$ and $x_1$, $x_2$, $y_1$, $y_2$ from $[0,1]$ such that

$$(x_1, t), \quad (x_2, 1-t), \quad (t, y_1) \text{ and } (1-t, y_2) \tag{3.4}$$

belong to $O$ for all $t \in ]0, \varrho[$. Loosely speaking, $O$ touches each boundary part of $R$ 'perpendicularly'. Any set $O$ with the above properties is called *round open set.*

**Definition 3.1** *Let $(p_1, \ldots, p_r)$ be a probability vector and let $(w_1, \ldots, w_r)$ be a set of diagonal affine contractions with a round open set. Then*

$$\mu := \langle w_1, \ldots, w_r; p_1, \ldots, p_r \rangle$$

*is called* Self-affine Multifractal, *for short* SAMF. *The characteristic values of $w_i$ will always be denoted by $\lambda_i$ and $\nu_i$.*

In order to compare $G_\delta$ with a suitable system of sets $V_{\underline{i}}$ the definition of $J_\delta$ (2.4) given in section 2.1 has to be modified: For any finite word $\underline{i} = i_1 \ldots i_n$ let

$$\lambda_{\underline{i}} := \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_n} \qquad \nu_{\underline{i}} := \nu_{i_1} \cdot \ldots \cdot \nu_{i_n}$$

as usually and define

$$\kappa(\underline{i}) := \min(\lambda_{\underline{i}}, \nu_{\underline{i}}) \geq \kappa(i_1 \ldots i_{n-1}) \cdot \kappa(i_n) \geq \kappa(i_1) \cdot \ldots \cdot \kappa(i_n).$$

Since $\kappa$ is only sub-multiplicative we prefer the slightly different notation and will not write $\kappa_{\underline{i}}$. Trivially $\kappa(\underline{i}) \leq \lambda \kappa(i_1 \ldots i_{n-1})$, thus $\kappa(\underline{i}) \downarrow 0$ ($|\underline{i}| \to \infty$) and the



construction (2.4) of $J_\delta$ on page 34 works with $\kappa(\underline{i})$, replacing $\lambda_{\underline{i}}$. For $0 < \delta < \nu$, the set $J_\delta$ is thus uniquely determined by

$$J_\delta = \{\underline{i} = i_1 \ldots i_n \in I \, : \, \kappa(\underline{i}) \leq \delta < \kappa(i_1 \ldots i_{n-1})\}. \tag{3.5}$$

Moreover, $J_\delta$ is tight and secure (see (2.5) and (2.6)) and

$$\nu\delta \leq \kappa(\underline{i}) \leq \delta \quad \text{for all } \underline{i} \in J_\delta.$$

Note that this definition of $J_\delta$ coincides with the one of section 2.1 when $\lambda_i = \nu_i$ ($i = 1, \ldots, r$).
The aim of this section is now to prove that it is enough to consider $V_{\underline{i}}$ ($\underline{i} \in J_\delta$) in order to determine $T(q)$. This is the essential step towards 'symbolic dynamics'. First, an estimate analogous to (2.21) is required, saying that a $\delta$-box is not intersected by too many sets $V_{\underline{i}}$ with $\underline{i} \in J_\delta$. It is only here, where the 'roundness' of $O$ is actually needed.

**Lemma 3.2** *Given two numbers $\rho_2 > \rho_1 > 0$, there is a number $b$ depending only on the affinities $w_1, \ldots, w_r$ such that*

$$\#J_{\delta'}(B) := \#\{\underline{i} \in J_{\delta'} \, : \, V_{\underline{i}} \cap (B)_1 \neq \emptyset\} \leq b$$

*for all $\delta > 0$ and $\delta' > 0$ with $\rho_2 > \delta'/\delta > \rho_1$ and for all $B \in G_\delta$.*

For reasons of simplicity the Cartesian product of two intervals of length $u$ and $v$ will be called a *u×v-rectangle*. It is not important whether a rectangle contains some parts of its boundary or not.
**Proof** Take $\underline{i} \in J_{\delta'}(B)$. The $\lambda_{\underline{i}} \times \nu_{\underline{i}}$-rectangle $R_{\underline{i}} := w_{\underline{i}}(R)$ contains $V_{\underline{i}}$, thus it must meet $(B)_1$.

**i)** Assume first that $\lambda_{\underline{i}} \geq \nu_{\underline{i}}$. Since $O$ is connected and bounded, there is a path within $O$ joining $(0, y_1)$ with $(1, y_2)$, which consists of finite many straight line segments $g_1, \ldots, g_N$, each one parallel to one of the axes. Choose $\varrho' > 0$ such that $U(\varrho', Q)$ is contained in $O$ for all endpoints $Q$ of the $g_i$, except $(0, y_1)$ and $(1, y_2)$. The path $g_1 \ldots g_N$ is by $w_{\underline{i}}$ mapped onto a path which crosses $R_{\underline{i}}$ parallel to the axes and which joins $w_{\underline{i}}(0, y_1)$ with $w_{\underline{i}}(1, y_2)$. Since $\nu_{\underline{i}} = \kappa(\underline{i}) \leq \delta' \leq \rho_2\delta$ and since $(B)_1$ meets the $\lambda_{\underline{i}} \times \nu_{\underline{i}}$-rectangle $R_{\underline{i}}$, there is $l$ such that $B' := (B)_{1+\varrho_2}$ intersects a part $h_{\underline{i}} := w_{\underline{i}}(g_l)$, which is parallel to the $x^{(1)}$-axis (see Fig. 3.1). By the 'roundness' of $O$ at least one endpoint of $h_{\underline{i}}$ must lie in $w_{\underline{i}}(O)$, the interior of $V_{\underline{i}}$. This point is denoted by $Q_{\underline{i}}$.

**i)a)** Assume first that $Q_{\underline{i}}$ lies in $B'$ (see the set $R_{\underline{i}}$ in figure 3.1). Then at least one quarter of the ball $U(\varrho'\nu_{\underline{i}}, Q_{\underline{i}})$ is contained in $B'$. Moreover, this ball is a subset of the ellipse $w_{\underline{i}}(U(\varrho', Q)$ and hence contained in $w_{\underline{i}}(O)$. Now the question is, how many such words exist. Since $w_{\underline{i}}(O)$ and $w_{\underline{j}}(O)$ do

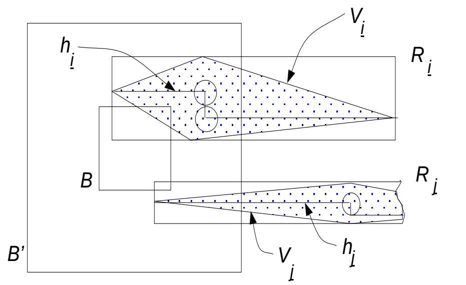


Figure 3.1: *Though the sets $R_{\underline{i}}$ do not have to be disjoint, they cannot overlap too much due to the horizontal paths $h_{\underline{i}}$.*

not intersect for $\underline{i} \neq \underline{j}$ ($J_{\delta'}$ is tight (2.5)), the just constructed balls $U$ are disjoint. Comparing volumes, there are at the most

$$b_1 := \frac{4}{\pi}\Big(\frac{3+2\rho_2}{\varrho'\nu\rho_1}\Big)^2$$

words of this kind in $J_{\delta'}$.

**i)b)** When $Q_{\underline{i}}$ lies outside $B'$ (see the set $R_{\underline{j}}$ in Fig. 3.1), then $h_{\underline{i}}$ must meet $\partial B'$ in one of its two vertical parts. Denote this intersection point by $S_{\underline{i}}$. Take two different words $\underline{i}$ and $\underline{j}$ satisfying case i)b) such that $S_{\underline{i}}$ and $S_{\underline{j}}$ lie on the same straight part of $\partial B'$. Then, the ball $U(\varrho'\nu_{\underline{i}}, Q_{\underline{i}})$ is disjoint with $V_{\underline{j}}$ and hence with $h_{\underline{j}}$, and vice versa. Thus, $S_{\underline{i}}$ and $S_{\underline{j}}$ are at least at distance $\varrho'\nu_{\underline{i}} \geq \varrho'\nu\delta' \geq \varrho'\nu\rho_1\delta$ of each other. Comparing the length of $\partial B'$ with this minimal distance proves that at the most

$$b_2 := 2\frac{3+2\rho_2}{\varrho'\nu\rho_1}$$

words of this kind are in $J_{\delta'}$.

**ii)** When $\lambda_{\underline{i}} < \nu_{\underline{i}}$, the same argumentation holds, showing that $b := 2(b_1 + b_2)$ is enough. $\diamond$

**Remark** From the proof it is immediate that $O$ can be allowed to have a finite number of connected components which all satisfy the 'roundness'-condition. The constant $b$ has then to be multiplied by the number of components.
As a consequence of the lemma:

**Corollary 3.1** *SAMFs are nonatomic.*



**Proof** The proof works exactly as the proof of lemma 2.7 except that (2.21) has to be replaced by lemma 3.2. $\diamond$
We continue with elementary properties of $\mu$: Denote the projection of $\mathbb{R}^2$ onto the $x^{(k)}$-axis by $\pi^{(k)}$ and set

$$w_i^{(1)}(x^{(1)}) := \vartheta_i\lambda_i x^{(1)} + u_i, \qquad w_i^{(2)}(x^{(2)}) := \zeta_i\nu_i x^{(2)} + v_i,$$

and

$$K^{(k)} := \pi^{(k)}(K), \qquad \mu^{(k)} := \pi^{(k)}{}_*\mu.$$

Then, $\mu^{(k)}$ is a multifractal on $\mathbb{R}$. Its singularity exponents and the other values relevant in the multifractal formalism will be marked with a '$^{(k)}$', i.e.

$$T^{(k)}(q) = (q-1)D_q{}^{(k)} = \limsup_{\delta\downarrow 0}\frac{\log\big(S_\delta^{(k)}(q)\big)}{-\log\delta}$$

and so on.

**Lemma 3.3**

$$\begin{aligned} \mu^{(k)} &= \langle w_1^{(k)}, \ldots, w_r^{(k)}; p_1, \ldots, p_r\rangle \\ K^{(k)} &= \langle w_1^{(k)}, \ldots, w_r^{(k)}\rangle = \mathrm{supp}(\mu^{(k)}). \end{aligned}$$

**Proof**

**i)** We show that $K^{(k)} = \langle w_1^{(k)}, \ldots, w_r^{(k)}\rangle$ for $k = 1$: If $x \in K^{(1)}$, then there is $y$ with $(x,y) \in K$. By the invariance of $K$ (2.8) there is an $i \in \{1, \ldots, r\}$ and $(x', y') \in K$ such that $(x,y) = w_i(x', y')$. In particular $x = w_i^{(1)}(x')$ since $w_i$ is diagonal. Thus

$$K^{(1)} \subset \bigcup_{i=1}^{r} w_i^{(1)}(K^{(1)}).$$

On the other hand, if $x = w_i^{(1)}(x')$ with $x' \in K^{(1)}$, then there is $y'$ such that $(x', y') \in K$ and $x = \pi^{(1)}(w_i(x', y')) \subset \pi^{(1)}(K) = K^{(1)}$. Since $K^{(1)}$ is compact this proves the claim.

**ii)** The diagonality of $w_i$ implies $\pi^{(k)} \circ w_i = w_i^{(k)} \circ \pi^{(k)}$. From this

$$\mu^{(k)} = \pi^{(k)}{}_*\mu = \sum_{i=1}^{r} p_i \pi^{(k)}{}_*(w_{i*}\mu) = \sum_{i=1}^{r} p_i w_i^{(k)}{}_*(\pi^{(k)}{}_*\mu) = \sum_{i=1}^{r} p_i w_i^{(k)}{}_*\mu^{(k)},$$

and $\mu^{(k)}$ is the unique invariant measure with support $K^{(k)}$ (see section 2.2). $\diamond$

After these preliminaries let us turn to $S_\delta(q)$. Let

$$\sigma(q,a,b,\gamma,J) := \sum_{\underline{i}\in J^+} p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^a \nu_{\underline{i}}{}^{\gamma-a} + \sum_{\underline{i}\in J^-} p_{\underline{i}}{}^q \nu_{\underline{i}}{}^b \lambda_{\underline{i}}{}^{\gamma-b} \tag{3.6}$$

for any finite set $J$ of finite words, where

$$J^+ := \{\underline{i}\in J \,:\, \lambda_{\underline{i}} > \nu_{\underline{i}}\} \qquad J^- := \{\underline{i}\in J \,:\, \lambda_{\underline{i}} \le \nu_{\underline{i}}\}. \tag{3.7}$$

This sum $\sigma$ will approximate $S_\delta(q)\cdot\delta^\gamma$ for $J=J_\delta$, $a=T^{(1)}(q)$ and $b=T^{(2)}(q)$ intuitively in the following way: consider

$$S^{(1)}_\varepsilon(q) = \sum_k \mu^{(1)}([k\varepsilon,(k+1)\varepsilon[)^q = \sum_k \mu([k\varepsilon,(k+1)\varepsilon[\times[0,1])^q \simeq \varepsilon^{-T^{(1)}(q)}. \tag{3.8}$$

For $\underline{i}\in J_\delta{}^+$ and $\varepsilon=\nu_{\underline{i}}/\lambda_{\underline{i}}\le 1$, the strips $[k\varepsilon,(k+1)\varepsilon[\times[0,1]$ are by $w_{\underline{i}}$ transformed into squares of side $\nu_{\underline{i}}\simeq\delta$, covering $V_{\underline{i}}$. The measure of such a square is roughly $p_{\underline{i}}\cdot\mu^{(1)}([k\varepsilon,(k+1)\varepsilon[)$. Doing similarly for $\underline{i}\in J_\delta{}^-$ and considering the obtained squares as an approximation of the $\delta$-boxes forming $G_\delta$, (3.8) leads to

$$\begin{aligned}
S_\delta(q) &\simeq \sum_{\underline{i}\in J_\delta{}^+}\sum_k \left(p_{\underline{i}}\cdot\mu^{(1)}([k\varepsilon,(k+1)\varepsilon[)\right)^q + \sum_{\underline{i}\in J_\delta{}^-}\sum_k \left(p_{\underline{i}}\cdot\mu^{(2)}([k\varepsilon,(k+1)\varepsilon[)\right)^q \\
&= \sum_{\underline{i}\in J_\delta{}^+} p_{\underline{i}}{}^q S^{(1)}_\varepsilon(q) + \sum_{\underline{i}\in J_\delta{}^-} p_{\underline{i}}{}^q S^{(2)}_\varepsilon(q) \\
&= \sum_{\underline{i}\in J_\delta{}^+} p_{\underline{i}}{}^q (\nu_{\underline{i}}/\lambda_{\underline{i}})^{-T^{(1)}(q)} + \sum_{\underline{i}\in J_\delta{}^-} p_{\underline{i}}{}^q (\lambda_{\underline{i}}/\nu_{\underline{i}})^{-T^{(2)}(q)}.
\end{aligned}$$

This partition sum will tend to zero for $q>1$ and to $\infty$ for $q<1$. To detect the power rate with respect to $\delta$ it is convenient to investigate

$$\begin{aligned}
S_\delta(q)\cdot\delta^\gamma &\simeq \sum_{\underline{i}\in J_\delta{}^+} p_{\underline{i}}{}^q (\nu_{\underline{i}}/\lambda_{\underline{i}})^{-T^{(1)}(q)}\cdot\nu_{\underline{i}}{}^\gamma + \sum_{\underline{i}\in J_\delta{}^-} p_{\underline{i}}{}^q (\lambda_{\underline{i}}/\nu_{\underline{i}})^{-T^{(2)}(q)}\cdot\lambda_{\underline{i}}{}^\gamma \\
&= \sigma(q, T^{(1)}(q), T^{(2)}(q), \gamma, J_\delta).
\end{aligned}$$

Step by step the above approximation will be made rigorous. The asymptotic behaviour of $\sigma$ is investigated in section 3.2.

**Lemma 3.4** *For any $q\in\mathbb{R}$, $k\in\{0,1\}$ and any $\overline{\gamma}>T^{(k)}(q)$, $\underline{\gamma}<\underline{T}^{(k)}(q)$ there is a number $c$ such that for all $\delta\in]0,1]$*

$$\frac{1}{c}S_\delta^{(k)}(q)\delta^{\overline{\gamma}} \le 1 \le c\cdot S_\delta^{(k)}(q)\delta^{\underline{\gamma}}$$

**Proof**

**i)** Remember that $-\infty<\underline{T}^{(k)}(q)$ in general and that $T^{(k)}(q)<\infty$ since the measures $\mu^{(k)}$ are contractive CMFs. Obviously there is a $\delta_0>0$ such that the assertion holds for $\delta<\delta_0$. Thus it is enough to prove that $S_\delta^{(k)}(q)$ is bounded from above and away from zero for $\delta\in[\delta_0,1]$. For the remainder let $\delta$ be from this interval.



**ii)** Take $q\ge 0$ first. Then $\mu^{(k)}((B)_1)^q\le 1$ and $S_\delta^{(k)}(q)\le \#G_\delta{}^{(k)}\le 1/\delta\le 1/\delta_0$. On the other hand, there is a $B$ with $\mu^{(k)}((B)_1)\ge\mu^{(k)}(B)\ge 1/\#G_\delta{}^{(k)}\ge\delta_0$, thus $S_\delta^{(k)}(q)\ge\delta_0{}^q$. Here $G_\delta{}^{(k)}$ denotes the set of all $\delta$-boxes on the $x^{(k)}$-axis with nonvanishing $\mu^{(k)}$-measure.

**iii)** Given $q\le 0$ it is obvious that $S_\delta^{(k)}(q)\ge 1$. For the upper bound note that $\mu^{(k)}$ is a CMF arising from an IFS—but not necessarily with a basic open set. However, $n$ can be chosen large enough to ensure $\lambda^n\le\delta_0$. Then the diameter of $V_{\underline{i}}{}^{(k)} := w_{\underline{i}}{}^{(k)}([0,1])$ is certainly smaller than $\delta_0$ provided $|\underline{i}|=n$. Given $B\in G_\delta{}^{(k)}$ there must be a word $\underline{i}$ of length $n$ such that $V_{\underline{i}}{}^{(k)}$ meets $B$ ($I_n$ is secure). Thus $V_{\underline{i}}{}^{(k)}$ is contained in $(B)_1$ and $\mu^{(k)}((B)_1)\ge p_{\underline{i}}$. Every $V_{\underline{i}}{}^{(k)}$ can meet at the most two boxes $B$. This implies $S_\delta^{(k)}(q)\le 2(p_1^q+\ldots+p_r^q)^n$. The independence of $n$ from $\delta$ completes the proof. ◇

**Lemma 3.5** *Given $q\ge 0$, $\eta>0$ and $\gamma\in\mathbb{R}$ there are numbers $c_1$ and $c_2$ such that for all $\delta>0$*

$$c_2\cdot\sigma(q,\underline{T}^{(1)}(q),\underline{T}^{(2)}(q),\gamma+\eta,J_\delta)\le S_\delta(q)\cdot\delta^\gamma\le c_1\cdot\sigma(q,T^{(1)}(q),T^{(2)}(q),\gamma-\eta,J_\delta).$$

**Proof** Let $\delta>0$ and $q\ge 0$.

**i)** The sets $V_{\underline{i}}$ are circumscribed by $\lambda_{\underline{i}}\times\nu_{\underline{i}}$-rectangles, most of which are long stretched and thin. For $\underline{i}\in J_\delta$ $V_{\underline{i}}$ will be subdivided into sets of diameter $\simeq\delta$. Take first the case $\lambda_{\underline{i}}>\nu_{\underline{i}}$, i.e. $\underline{i}\in J_\delta{}^+$. Set

$$\varepsilon_{\underline{i}} := \frac{\nu_{\underline{i}}}{\lambda_{\underline{i}}}.$$

Then $\varepsilon_{\underline{i}}\le 1$. Define

$$C(k,\underline{i}) := [(k-1)\cdot\varepsilon_{\underline{i}}, k\cdot\varepsilon_{\underline{i}}[\times[0,1] \tag{3.9}$$

for $k=1,\ldots,\lceil 1/\varepsilon_{\underline{i}}\rceil$, where $\lceil x\rceil$ is the smallest integer greater than or equal to $x$. The sets $D(k,\underline{i}) := w_{\underline{i}}(C(k,\underline{i}))$ constitute a disjoint covering of $V_{\underline{i}}$ by $\varepsilon_{\underline{i}}\lambda_{\underline{i}}\times\nu_{\underline{i}}$-rectangles, which are in fact squares (see Fig. 3.2). From lemma 3.3 follows

$$S_{\varepsilon_{\underline{i}}}{}^{(1)}(q) = \sum_{\mu(C(k,\underline{i}))\neq 0}(\pi^{(1)}{}_*\mu([(k-2)\cdot\varepsilon_{\underline{i}},(k+1)\cdot\varepsilon_{\underline{i}}[))^q = \sum_{\mu(C(k,\underline{i}))\neq 0}\mu((C(k,\underline{i}))_1)^q,$$

where $(C(k,\underline{i}))_1 := [(k-1)\cdot\varepsilon_{\underline{i}},(k+2)\cdot\varepsilon_{\underline{i}}[\times[0,1]$. In the case $\lambda_{\underline{i}}\le\nu_{\underline{i}}$ one finds of course

$$\sum_{\mu(C(k,\underline{i}))\neq 0}\mu((C(k,\underline{i}))_1)^q = S_{\varepsilon_{\underline{i}}}{}^{(2)}(q)$$

with the obvious modifications.

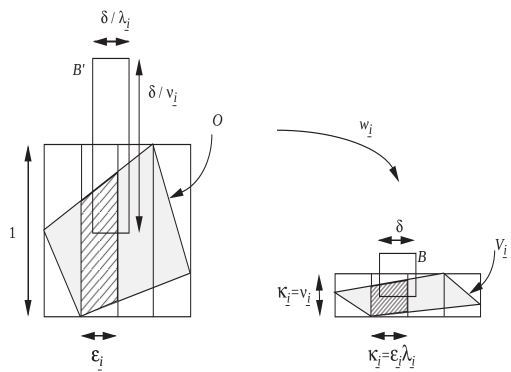


Figure 3.2: *The square* $[0,1]^2$ *is subdivided into strips* $C(k,\underline{i})$ *of width* $\varepsilon_{\underline{i}}$*, which are mapped onto squares of side* $\kappa(\underline{i}) \simeq \delta$.

**ii)** Take $B \in G_\delta$. Observing that $J_\delta$ is secure and tight, and that the support of $\mu$ is contained in $\overline{O}$, the invariance of $\mu$ yields:

$$\begin{aligned}\mu((B)_1) &= \sum_{\underline{i}\in J_\delta} p_{\underline{i}}\mu\Big(w_{\underline{i}}^{-1}((B)_1 \cap V_{\underline{i}})\Big)\\ &= \sum_{\underline{i}\in J_\delta}\sum_{k=1}^{\lceil 1/\varepsilon_{\underline{i}}\rceil} p_{\underline{i}}\mu\Big(w_{\underline{i}}^{-1}((B)_1 \cap D(k,\underline{i}) \cap V_{\underline{i}})\Big)\\ &\le \sum_{\underline{i}\in J_\delta}{\sum}' p_{\underline{i}}\mu(C(k,\underline{i})).\end{aligned}$$

Here, $\sum'$ runs for fixed $\underline{i}$ over all $k$ such that $(B)_1$, $D(k,\underline{i})$ and $V_{\underline{i}}$ intersect. Since the $D(k,\underline{i})$ are disjoint and contain squares of side $\delta\nu$, there are for each $\underline{i}$ at the most $1+3/\nu$ such integers $k$. Moreover, $(B)_1$ meets only $b$ sets $V_{\underline{i}}$ (lemma 3.2). Consequently the last double sum has actually at the most $b_1 := b(1+3/\nu)$ terms. So it is possible to find $k = k(B)$ and $\underline{i} = \underline{i}(B)$ such that $D(k(B),\underline{i}(B))$ meets $(B)_1$ and such that for $C(B) := C(k(B),\underline{i}(B))$ the inequality

$$0 \neq \mu((B)_1) \le b_1 \cdot p_{\underline{i}(B)}\mu(C(B))$$

holds. This leads to

$$S_\delta(q) = \sum_{B\in G_\delta}\mu((B)_1)^q \le b_1^q \sum_{B\in G_\delta} p^q_{\underline{i}(B)}\mu\Big((C(B))_1\Big)^q.$$

Since every fixed $D(k,\underline{i})$ is a square of side $\kappa(\underline{i}) \le \delta$, it can at the most meet 16 sets $(B)_1$ with $B$ from $G_\delta$. Thus at the most 16 pairs $(k(B),\underline{i}(B))$ can coincide and hence



$$\begin{aligned}S_\delta(q) &\le 16{b_1}^q \sum_{\underline{i}\in J_\delta}\sum_{\mu(C(k,\underline{i}))\neq 0} {p_{\underline{i}}}^q\mu((C(k,\underline{i}))_1)^q\\ &= 16{b_1}^q\Big(\sum_{\underline{i}\in {J_\delta}^+} {p_{\underline{i}}}^q {S_{\varepsilon_{\underline{i}}}}^{(1)}(q) + \sum_{\underline{i}\in {J_\delta}^-} {p_{\underline{i}}}^q {S_{\varepsilon_{\underline{i}}}}^{(2)}(q)\Big).\end{aligned}$$

Now choose $\overline{\gamma}_k = T^{(k)}(q) + \eta$ $(k=1,2)$. Setting $c' = 16c{b_1}^q$ with $c$ from lemma 3.4 gives

$$S_\delta(q) \le c'\Big(\sum_{\underline{i}\in {J_\delta}^+} {p_{\underline{i}}}^q(\lambda_{\underline{i}}/\nu_{\underline{i}})^{\overline{\gamma}_1} + \sum_{\underline{i}\in {J_\delta}^-} {p_{\underline{i}}}^q(\nu_{\underline{i}}/\lambda_{\underline{i}})^{\overline{\gamma}_2}\Big).$$

Writing $a = T^{(1)}(q)$, $b = T^{(2)}(q)$, $c_1 = c'\max(1,\nu^{-\gamma})$ for short and observing $\kappa(\underline{i}) \le \delta \le \kappa(\underline{i})/\nu$ yields

$$\begin{aligned}S_\delta(q)\delta^\gamma &\le c_1\Big(\sum_{\underline{i}\in {J_\delta}^+} {p_{\underline{i}}}^q{\lambda_{\underline{i}}}^{\overline{\gamma}_1}{\nu_{\underline{i}}}^{\gamma-\overline{\gamma}_1} + \sum_{\underline{i}\in {J_\delta}^-} {p_{\underline{i}}}^q{\nu_{\underline{i}}}^{\overline{\gamma}_2}{\lambda_{\underline{i}}}^{\gamma-\overline{\gamma}_2}\Big)\\ &= c_1\Big(\sum_{\underline{i}\in {J_\delta}^+} {\lambda_{\underline{i}}}^\eta\cdot{p_{\underline{i}}}^q{\lambda_{\underline{i}}}^a{\nu_{\underline{i}}}^{\gamma-\eta-a} + \sum_{\underline{i}\in {J_\delta}^-} {\nu_{\underline{i}}}^\eta\cdot{p_{\underline{i}}}^q{\nu_{\underline{i}}}^b{\lambda_{\underline{i}}}^{\gamma-\eta-b}\Big)\\ &\le c_1\sigma(q,a,b,\gamma-\eta,J_\delta).\end{aligned}$$

The last estimate used $\lambda < 1$ (3.2) and $\eta > 0$.

**iii)** The same argumentation as above but with $B$ and $D(k,\underline{i})$ interchanged provides the desired lower bound. So, take $\underline{i} \in J_\delta$ and $k$ such that $\mu(C(k,\underline{i})) \neq 0$. The set $D_1(k,\underline{i}) := w_{\underline{i}}((C(k,\underline{i}))_1) \cap V_{\underline{i}}$ is contained in a rectangle with sides $\kappa(\underline{i})$ and $3\kappa(\underline{i})$. Thus there are at most eight boxes from $G_\delta$ meeting $D_1(k,\underline{i})$ and hence one among them, say $B(k,\underline{i})$, with

$$\mu(D_1(k,\underline{i})) \le 8\cdot\mu(B(k,\underline{i})).$$

Using the invariance of $\mu$ this allows the estimate

$$8^q\mu((B(k,\underline{i}))_1)^q \ge \mu(D_1(k,\underline{i}))^q \ge {p_{\underline{i}}}^q\mu((C(k,\underline{i}))_1)^q.$$

On the other hand, any fixed $B \in G_\delta$ can only meet $b$ sets $V_{\underline{i}}$. And for each such $\underline{i}$ the box $B$ can only intersect $(3+1/\nu)$ sets $D_1(k,\underline{i})$, since ${w_{\underline{i}}}^{-1}(B)$ has to intersect $(C(k,\underline{i}))_1$. Thus, at the most $b_2 := b(3+1/\nu)$ sets $B(k,\underline{i})$ can coincide with $B$. This leads to

$$\begin{aligned}b_2 8^q S_\delta(q) &\ge \sum_{\underline{i}\in J_\delta}\sum_{\mu(C(k,\underline{i}))\neq 0} {p_{\underline{i}}}^q\mu((C(k,\underline{i}))_1)^q\\ &= \sum_{\underline{i}\in {J_\delta}^+} {p_{\underline{i}}}^q {S_{\varepsilon_{\underline{i}}}}^{(1)}(q) + \sum_{\underline{i}\in {J_\delta}^-} {p_{\underline{i}}}^q {S_{\varepsilon_{\underline{i}}}}^{(2)}(q).\end{aligned}$$

Now choose $\underline{\gamma}_k = \underline{T}^{(k)}(q) - \eta$ $(k=1,2)$. With $c'' = 8^q c b_2$ lemma 3.4 gives

$$c'' S_\delta(q) \ge \Big(\sum_{\underline{i}\in {J_\delta}^+} {p_{\underline{i}}}^q(\lambda_{\underline{i}}/\nu_{\underline{i}})^{\underline{\gamma}_1} + \sum_{\underline{i}\in {J_\delta}^-} {p_{\underline{i}}}^q(\nu_{\underline{i}}/\lambda_{\underline{i}})^{\underline{\gamma}_2}\Big).$$

Writing $a = \underline{T}^{(1)}(q)$, $b = \underline{T}^{(2)}(q)$, $c_2 = \min(1, \nu^{-\gamma})/c''$ for short and observing $\kappa(\underline{i}) \le \delta \le \kappa(\underline{i})/\nu$ leads to

$$\begin{aligned} S_\delta(q)\delta^\gamma &\ge c_2\Big( \sum_{\underline{i}\in J_\delta{}^+} p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^{\underline{\gamma}_1} \nu_{\underline{i}}{}^{\gamma-\underline{\gamma}_1} + \sum_{\underline{i}\in J_\delta{}^-} p_{\underline{i}}{}^q \nu_{\underline{i}}{}^{\underline{\gamma}_2} \lambda_{\underline{i}}{}^{\gamma-\underline{\gamma}_2}\Big) \\ &= c_2\Big( \sum_{\underline{i}\in J_\delta{}^+} \lambda_{\underline{i}}{}^{-\eta} \cdot p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^a \nu_{\underline{i}}{}^{\gamma+\eta-a} + \sum_{\underline{i}\in J_\delta{}^-} \nu_{\underline{i}}{}^{-\eta} \cdot p_{\underline{i}}{}^q \nu_{\underline{i}}{}^b \lambda_{\underline{i}}{}^{\gamma+\eta-b}\Big) \\ &\ge c_1\sigma(q, a, b, \gamma+\eta, J_\delta). \end{aligned}$$

$\diamondsuit$

**Lemma 3.6** *Given $q < 0$, $\eta > 0$ and $\gamma \in \mathbb{R}$ there is a number $c_3$ such that for all $\delta > 0$*

$$S_\delta(q) \cdot \delta^\gamma \le c_3 \cdot \sigma(q, T^{(1)}(q), T^{(2)}(q), \gamma - \eta, J_{\delta/3}).$$

**Proof** The notation of the proof of lemma 3.5 is kept in use. Let $q < 0$, $\delta > 0$ and set $\delta' = \delta/3$. Take $B \in G_\delta$. Since $J_{\delta'}$ is secure there is an integer $k$ and a word $\underline{i} \in J_{\delta'}$ such that for $C(B) := C(k, \underline{i})$

$$0 \ne \mu(w_{\underline{i}}{}^{-1}(B) \cap C(B)) \le \mu(C(B)).$$

This implies in particular that $B' = w_{\underline{i}}{}^{-1}((B)_1)$ contains $(C(B))_1$: considering the case $\lambda_{\underline{i}} > \nu_{\underline{i}}$ first, $B'$ is a $3\delta/\lambda_{\underline{i}} \times 3\delta/\nu_{\underline{i}}$-rectangle with $w_{\underline{i}}{}^{-1}(B)$ concentric in its middle. Moreover, $\delta/\lambda_{\underline{i}} = 3\delta'/\lambda_{\underline{i}} = 3\varepsilon_{\underline{i}}\delta'/\nu_{\underline{i}} \ge 3\varepsilon_{\underline{i}}$ and $\delta/\nu_{\underline{i}} = 3\delta'/\nu_{\underline{i}} \ge 3$. Since $w_{\underline{i}}{}^{-1}(B)$ and $C(B)$ intersect the claim follows. It is here where the idea of the new formalism enters, which says to use $\mu((B)_1)$ of boxes $B$ with nonvanishing measure. Similar for $\lambda_{\underline{i}} \le \nu_{\underline{i}}$. As a consequence of the invariance of $\mu$

$$\mu((B)_1) \ge p_{\underline{i}}\mu\Big(w_{\underline{i}}{}^{-1}((B)_1)\Big) \ge p_{\underline{i}}\mu\Big((C(B))_1\Big) \ne 0.$$

Since $\delta' < \delta$ there are at the most four boxes $B$ with coinciding pair $(k, \underline{i})$ and thus

$$S_\delta(q) \le \sum_{B\in G_\delta} p^q_{\underline{i}(B)}\mu\Big((C(B))_1\Big)^q \le 4\sum_{\underline{i}\in J_{\delta'}} \sum_{\mu(C(k,\underline{i}))\ne 0} p_{\underline{i}}{}^q \mu\Big((C(k,\underline{i}))_1\Big)^q.$$

The rest of the proof is essentially a repetition of the argument at the end of step ii) in the proof of lemma 3.5. $\diamondsuit$

For negative $q$ it is not so easy to derive a lower bound of $S_\delta(q)$. Even more information is required about the geometry of $\mu$: similar as in the self-similar case $\mu$ should not concentrate at the boundary of the open set.

To be more precise: an IFS $(w_1, \ldots, w_r)$ of diagonal affine contractions will be called *vertically centered* if there is an open set $O = ]s, t[ \times ]u, v[$ such that for any $x \in K^{(1)}$ there is a number $l$ in $I_1$ such that $x \in \pi^{(1)}(V_l)$ and $\pi^{(2)}(V_l) \subset ]u, v[$. Note that $V_l$ is a closed set. Mutatis mutandis *horizontally centered* is defined.



Figure 3.3 provides a multifractal with centered IFS. For such measures a lower bound of $S_\delta(q)$ can be given also for negative $q$.

However, it is an important note that the IFS generating a multifractal is not unique. In applications it can be helpful to change to a coarser construction of the same invariant set. More precisely speaking, the invariance (2.8) of $K$ holds for any secure and tight, finite set $J$ of words, not only for $I_1$. The resulting codespace $I^*_\infty = J^{\mathbb{N}}$ can naturally be identified with $I_\infty = I_1^{\mathbb{N}}$, which motivates the term *coarser IFS*. Supplied with the obvious product measure $J^{\mathbb{N}}$ will produce the same SAMF $\mu$, since the invariance (2.10) of $\mu$ holds also with $J$. Summa summarum the following definition is effective:

**Definition 3.7** *A SAMF $\mu = \langle w_1, \ldots, w_r; p_1, \ldots, p_r\rangle$ will be called* centered self-affine multifractal, *for short* C-SAMF, *if it possesses two coarser IFS, one centered vertically and one centered horizontally.*

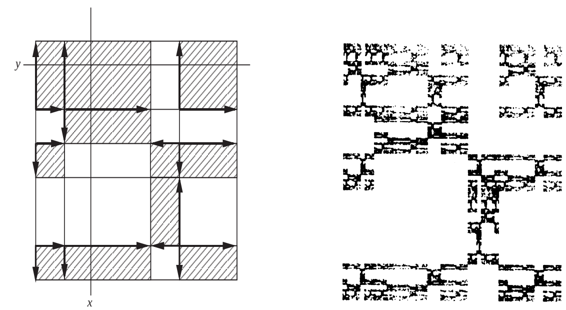


Figure 3.3: *On the left the construction of a SAMF: The unit square $\overline{O}$ is drawn as well as its images $w_i(\overline{O})$, marked by the shaded regions. The arrows reveal the induced orientation. Through this data, the IFS is uniquely determined and obviously it is centered. Moreover, the projections of the invariant set $K$ are even self-similar, allowing a calculation of the spectrum as described in subsection 3.5.1. On the right the multifractal corresponding to the probabilities $p_1 = p_2 = p_5 = p_7 = p_{10} = p_{11} = 1/25$, $p_3 = p_4 = p_6 = p_8 = p_9$, numbering the maps from the left to the right and from the bottom to the top. The image is composed of 150'000 points provided by a random algorithm.*

The strong condition ‘centered’ is needed to be sure that the measure $\mu$ is nowhere concentrated on the boundary of the open set. This corresponds to the method of the new formalism using the measures $\mu((B)_1)$ of boxes with $\mu(B) \ne 0$. More general assumptions become apparent in lemma 3.9 ii).

In examples, the situation of the lemma below is often encountered.

**Lemma 3.8 (Self-Similar Projections)** *Let $\mu$ be a SAMF with round open set $]0,1[^2$. Assume that $w_i^{(k)}(O) \cap w_j^{(k)}(O) \neq \emptyset$ implies $w_i^{(k)} = w_j^{(k)}$. If for each $i$*

$$\pi^{(k)}(V_i) \cap \{0,1\} \neq \emptyset \quad \Rightarrow \quad \exists j \neq i : \pi^{(3-k)}(V_i) = \pi^{(3-k)}(V_j),$$

*then the coarser IFS $\{w_{ij} : ij \in I_2\}$ is centered, and so is $\mu$. Moreover, the projections of the measure $\mu^{(k)}$ are then self-similar.*

To speak in pictures: sufficient for $\mu$ to be centered are the following three conditions.

- The projections $K^{(k)}$ (lemma 3.3) are self-similar. (The sets $V_i$ are then arranged in rows and columns.)
- If a column contains only one $V_i$, then $V_i$ is not allowed to touch the 'bottom' or the 'top'.
- Similar for rows with only one entry.

See Fig. 3.4 or 3.13.
**Proof** Mark the words of the coarser codespace $I_2^{\mathbb{N}} =: I_\infty^*$ by a $^*$.

**i)** In order to establish $I_\infty^*$ as vertically centered take $x \in K^{(1)}$. There is $y \in [0,1]$ and $ij \in I_2$ with $(x,y) \in V_{ij}$. If $\pi^{(2)}(V_{ij})$ contains neither 0 nor 1, then $l^* = ij$ is a possible choice. Otherwise $\pi^{(2)}(V_j)$ must intersect $\{0,1\}$. By assumption there is $k \neq j$ with $\pi^{(1)}(V_k) = \pi^{(1)}(V_j)$. Consider the set $V_{ik}$. It has the same $\pi^{(1)}$-projection as $V_{ij}$ (see Fig. 3.4) and hence contains $x$. Furthermore, $\pi^{(2)}(V_{ij}) \neq \pi^{(2)}(V_{ik}) \subset \pi^{(2)}(V_i)$ implies that its $\pi^{(2)}$-projection contains neither 0 nor 1. So it is enough to choose $l^* = ik$.

**ii)** The case $y \in K^{(2)}$ is treated the same way. The self-similarity of $\mu^{(k)}$ is shown in subsection 3.5.1. $\diamond$

**Lemma 3.9** *For any C-SAMF $\mu$, any $q < 0$, $\eta > 0$ and $\gamma \in \mathbb{R}$ there are numbers $c_4$, $c_5$ and $\Theta > 0$ such that for all $\delta > 0$ and $\delta' = c_5 \cdot \delta$*

$$S_\delta(q)\delta^{\gamma-\eta} \geq c_4 \sigma(q, \underline{T}^{(1)}(q), \underline{T}^{(2)}(q), \gamma, J_{\delta'}).$$

**Proof**

**i)** Take $\underline{i}$ from $J_{\delta'}^+$. By assumption there is a vertically centered, coarser IFS of $\mu$ with codespace $I_\infty^* = J^{\mathbb{N}}$ and round open set $O^* = ]s,t[\times]u,v[$. Since $J$ is secure, the fixpoints of $w_1, \ldots, w_r$ must lie inside $O^*$, and therefore, $O^*$ satisfies the OSC also for the basic IFS $w_1, \ldots, w_r$. For reasons which will become transparent only in iv), we choose $O = O^*$. This means that not only the codespaces $I_\infty$ and $I_\infty^*$ can naturally be identified but also the construction of $\mu$ by cylindrical sets, due to $V_{\underline{i}} = w_{\underline{i}}(\overline{O^*})$. Similar as in (3.9) set

$$C(k, \underline{i}) := [(k-1) \cdot \varepsilon_{\underline{i}}, k \cdot \varepsilon_{\underline{i}}[ \times [u,v]$$



with the $k$-range $\lceil s/\varepsilon_{\underline{i}} \rceil + 1, \ldots, \lceil t/\varepsilon_{\underline{i}} \rceil$. Take an integer $k$ with $\mu(C(k,\underline{i})) \neq 0$. The idea is to estimate $\mu((C(k,\underline{i}))_1)$ from below in the following manner:

$$p_{\underline{i}}\mu((C(k,\underline{i}))_1) \geq p_{\underline{i}}\mu({w_{\underline{i}}}^{-1}((B)_1)) = \sum_{\underline{j} \in J_\delta} p_{\underline{j}}\mu({w_{\underline{j}}}^{-1}((B)_1)) = \mu((B)_1),$$

where $B$ is from $G_\delta$. Therefore, ${w_{\underline{i}}}^{-1}((B)_1)$ is required to be a subset of $(C(k,\underline{i}))_1$ and $(B)_1$ to be contained in $\text{int}(V_{\underline{i}})$. Once such a box $B$ is found, it is immediate that it can at the most belong to three different $C(k,\underline{i})$: $B$ lies in $\text{int}(V_{\underline{i}})$, which are mutually disjoint for different $\underline{i}$ from $J_{\delta'}$, and ${w_{\underline{i}}}^{-1}((B)_1)$ is a subset of $(C(k,\underline{i}))_1$. With $\underline{\gamma}_1 = \underline{T}^{(1)}(q) - \eta$ this results in

$$3S_\delta(q) \geq \sum_{\underline{i} \in J_{\delta'}^+} \sum_{\mu(C(k,\underline{i})) \neq 0} {p_{\underline{i}}}^q \mu((C(k,\underline{i}))_1)^q = \sum_{\underline{i} \in J_{\delta'}^+} {p_{\underline{i}}}^q {S_{\varepsilon_{\underline{i}}}}^{(1)}(q)$$

and with lemma 3.4

$$S_\delta(q) \geq c/3 \cdot \sum_{\underline{i} \in J_{\delta'}^+} {p_{\underline{i}}}^q (\lambda_{\underline{i}}/\nu_{\underline{i}})^{\underline{\gamma}_1}.$$

Note that this last relation is completely independent of the choice of $O$. The same procedure applied to words $\underline{i} \in J_{\delta'}^-$, with possibly different codespace $I_\infty^{**}$ and round open set $O = O^{**}$, yields

$$2 \cdot S_\delta(q) \geq (c/3 + c'/3) \cdot \Big( \sum_{\underline{i} \in J_{\delta'}^+} {p_{\underline{i}}}^q (\lambda_{\underline{i}}/\nu_{\underline{i}})^{\underline{\gamma}_1} + \sum_{\underline{i} \in J_{\delta'}^-} {p_{\underline{i}}}^q (\nu_{\underline{i}}/\lambda_{\underline{i}})^{\underline{\gamma}_2} \Big).$$

Then the rest of the proof will follow the lines of lemma 3.5 iii).

**ii)** It remains only to find $B$. To keep ideas clear, fix $\underline{i} \in J_{\delta'}^+$ and $k$ with $\mu(C(k,\underline{i})) \neq 0$ for the rest of the proof. The arguments given will be symmetric in order to cover the case $\underline{i} \in J_{\delta'}^-$ as well. The remaining task is of purely geometric nature. It concerns only the position of a certain box $B$. In particular using now the codespace $I_\infty^*$ will not affect step i). The various codespaces are distinguished by asterisks ($^*$).
Remember that $I_\infty^*$ is vertically and $I_\infty^{**}$ horizontally centered. As an immediate consequence it is possible to choose $\Theta > 0$ such that

$$\begin{aligned} x \in K^{(1)} &\Rightarrow \exists l^* : x \in \pi^{(1)}(V_{l^*}^*) \text{ and } \pi^{(2)}(V_{l^*}^*) \subset ]u+\Theta, v-\Theta[ \\ y \in K^{(2)} &\Rightarrow \exists m^{**} : y \in \pi^{(2)}(V_{m^{**}}^{**}) \text{ and } \pi^{(1)}(V_{m^{**}}^{**}) \subset ]s+\Theta, t-\Theta[ \end{aligned}$$

This means that $\mu$ is nowhere concentrated at the boundary of $O$. To be more precise: every strip $[x,x'] \times \mathbb{R}$ with nonvanishing measure meets the support $K$ certainly at distance at least $\Theta$ from the 'upper' and the 'lower' boundary of $\overline{O}$ (see Fig. 3.4). This is what is actually needed for the proof.

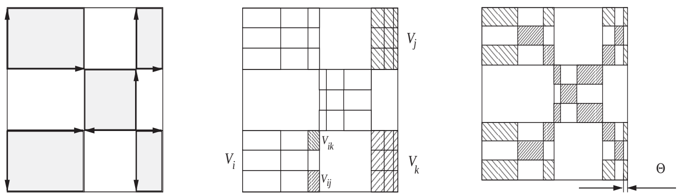


Figure 3.4: *A SAMF with $r = 5$ to which lemma 3.8 applies: consider the second iterates. To every set $V_{ij}$ touching the 'bottom' of $O$ there is a $V_{ik}$ in the same column at distance greater or equal $\Theta$ from the bottom as well as from the top. Thus this SAMF is centered and concentrates nowhere on the boundary of $O$. (See step ii) in the proof of lemma 3.9.)*

**iii)** Finally, some control in the direction complementary to ii) is required. Without loss of generality and after eventually choosing a smaller $\Theta$, there is a letter $m^*$ from $I_1^*$ such that

$$\pi^{(1)}(V^*_{m^*}) \subset ]s+\Theta, t-\Theta[$$

and similar for $I^{**}_\infty$. For, if there were no such $m^*$, then all $\pi^{(1)}(V^*_{m^*})$ would contain either $s$ or $t$. Then two cases would be possible. First, there could be $m^* \neq k^*$ such that $\pi^{(1)}(V^*_{m^*})$ contained $s$ and $\pi^{(1)}(V^*_{k^*})$ contained $t$. Then at least one of the words $m^*m^*$, $m^*k^*$, $k^*m^*$, $k^*k^*$ would satisfy the condition above, and the still centered IFS corresponding to $(I_2^*)^{\mathbb{N}}$ could be used. Otherwise, $\mu$ would equal a translate of $\mu^{(2)}$ and could be considered as a self-similar multifractal. This case will not be treated here since it is trivial. An easy check shows that the conclusions drawn from this lemma (theorem 3.3) hold certainly also in this case.

**iv)** At last $B$ will be constructed. For didactical reasons the definition of $c_5$ is postponed. Since $\mu(C(k,\underline{i})) \neq 0$, there is $V^*_{l^*}$ which meets $C(k,\underline{i})$ in a point, say $(x,y)$, of $K$. Without loss of generality $\pi^{(2)}(V^*_{l^*}) \subset ]u+\Theta, v-\Theta[$ by ii).
Now consider a code $\underline{k}^*_\infty$ of $(x,y)$ starting with $l^*$. In particular $(x,y) = \pi^*(\underline{k}^*_\infty)$ and $k_1^* = l^*$. Then there is a unique $n$ with

$$\lambda^*_{k_1^*\ldots k_n^*} \leq \varepsilon_{\underline{i}} \cdot \underline{\nu}/2 < \lambda_{k_1^*\ldots k_{n-1}^*}.$$

Thereby $\underline{\nu}$ denotes the minimum of the $\nu^*$ (3.2) of the two centered IFS involved. The factor $\underline{\nu}/2$ is needed to cover the case $\varepsilon_{\underline{i}} = 1$, which may well occur. Choose a word $m^*$ from $I_1^*$ according to iii) such that $\pi^{(1)}(V^*_{m^*})$ is contained in $]s+\Theta, t-\Theta[$. Let $\underline{k}^* := k_1^* \ldots k_n^*$ and $\underline{j}^* := \underline{k}^* * m^*$. Then the set $V^*_{\underline{k}^*}$ meets $C(k,\underline{i})$ in $(x,y)$ and is suitably small to be still in 'the middle' of $(C(k,\underline{i}))_1$. Moreover, its subset $V^*_{\underline{j}^*}$ is at a convenient distance of the border of $O = O^*$: it must lie in $[s+\lambda^*_{\underline{k}^*}\Theta, t-\lambda^*_{\underline{k}^*}\Theta] \times [u+\Theta, v-\Theta]$ since it is the



image of $V^*_{m^*}$ under $w^*_{\underline{k}^*}$ and since it is a subset of $V^*_{l^*}$ (see Fig. 3.5).
Now consider $B' = w_{\underline{i}}^{-1}(B)$, where $B$ varies over all $\delta$-boxes. Since the sets $B'$

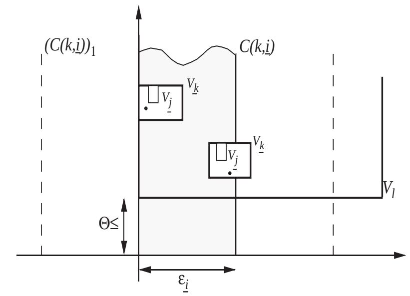


Figure 3.5: *This picture shows the lower part of $C(k,\underline{i})$ for $k = 0$ and reveals the construction of $V_{\underline{j}}$ for two different points $(x,y)$ of $K$ (the black dots). Note that $V_{\underline{j}}$ must lie at 'great' distance from the boundaries of $(C(k,\underline{i}))_1$ and of $O$, in order to know that $B_1'$—which meets $V_{\underline{j}}$—is indeed a subset of the latter two. (See the proof of lemma 3.9.)*

cover the plane there is one with nonvanishing measure which meets $V^*_{\underline{j}^*}$. The special choice $O = O^*$ in step i) implies, as will be shown straight away, that $B_1' = w_{\underline{i}}^{-1}((B)_1)$ is contained in $(C(k,\underline{i}))_1$ as well as in $O$. With this proven one concludes immediately $B \subset (B)_1 \subset \mathrm{int}(V_{\underline{i}})$, thus $\mu(B) = p_{\underline{i}}\mu(B') \neq 0$ and $B$ belongs indeed to $G_\delta$ and is the box desired in i).
First note that $B_1'$ is a $(3\delta/\lambda_{\underline{i}} \times 3\delta/\nu_{\underline{i}})$-rectangle concentric to $B'$. Choosing finally $c_5 = 6/\Theta\nu(\underline{\nu})^2$ gives

$$\frac{3\delta}{\lambda_{\underline{i}}} = \frac{\Theta\nu\underline{\nu}^2\delta'}{2\lambda_{\underline{i}}} \leq \frac{\Theta\underline{\nu}^2}{2}\frac{\nu_{\underline{i}}}{\lambda_{\underline{i}}} = \Theta\frac{\underline{\nu}^2\varepsilon_{\underline{i}}}{2} < \min(\lambda^*_{\underline{k}^*}\Theta, \varepsilon_{\underline{i}}/2).$$

Moreover, $\mathrm{dist}(\pi^{(1)}(B_1'), x) \leq \mathrm{diam}(\pi^{(1)}(V^*_{\underline{k}^*})) = \lambda^*_{\underline{k}^*} \leq \varepsilon_{\underline{i}}/2$ and thus $\pi^{(1)}(B_1')$ is a subset of $\pi^{(1)}((C(k,\underline{i}))_1)$ (which is not necessarily contained in $]s,t[$). On the other hand, $\mathrm{dist}(\pi^{(1)}(V^*_{\underline{j}^*}), \{s,t\}) \geq \lambda^*_{\underline{k}^*}\Theta$ and so $\pi^{(1)}(B_1')$ lies in $]s,t[$. Secondly,

$$\frac{3\delta}{\nu_{\underline{i}}} = \frac{\Theta\nu\underline{\nu}^2\delta'}{2\nu_{\underline{i}}} \leq \frac{\Theta\underline{\nu}^2}{2} < \Theta,$$

and $\mathrm{dist}(\pi^{(2)}(V^*_{\underline{j}^*}), \{u,v\}) \geq \Theta$. Thus $\pi^{(2)}(B_1')$ is contained in $]u,v[$. Summarizing, $B_1'$ is indeed a subset of $(C(k,\underline{i}))_1$ and of $O$. $\diamond$

## 3.2 Limit Theorems

This section is devoted to the asymptotic behaviour of $\sigma(q,a,b,\gamma,J_\delta)$ for $\delta \to 0$ with focus in its dependence on $\gamma$. This is purely a question of convergence.
First we consider two trivial cases.

**Lemma 3.10** *Let $J$ be an arbitrary set of words. Then*

$$\sigma(q,a,b,\gamma,J) = \begin{cases} \sum_{\underline{i}\in J} p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^a \nu_{\underline{i}}{}^{\gamma-a} & \text{if } \lambda_{\underline{i}} \ge \nu_{\underline{i}} \ (i=1,\dots,r)\ , \\ \sum_{\underline{i}\in J} p_{\underline{i}}{}^q \nu_{\underline{i}}{}^b \lambda_{\underline{i}}{}^{\gamma-b} & \text{if } \lambda_{\underline{i}} \le \nu_{\underline{i}} \ (i=1,\dots,r)\ . \end{cases}$$

**Proof** Obviously $\lambda_{\underline{i}} = \nu_{\underline{i}}$ implies $\lambda_{\underline{i}}{}^a \nu_{\underline{i}}{}^{\gamma-a} = \nu_{\underline{i}}{}^\gamma = \nu_{\underline{i}}{}^b \lambda_{\underline{i}}{}^{\gamma-b}$. $\diamond$
Assume for the moment that $\lambda_{\underline{i}} \ge \nu_{\underline{i}}$ $(i=1,\dots,r)$ and that $\gamma$ is chosen such that

$$\sum_{i=1}^r p_i{}^q \lambda_i{}^a \nu_i{}^{\gamma-a} = 1.$$

Then $\sigma(q,a,b,\gamma,J_\delta) = 1$ for all $\delta > 0$, since $J_\delta$ is secure and tight, and the asymptotics are indeed trivial.
If one of the cases of lemma 3.10 applies, we call the corresponding SAMF *ordered*.
For the remainder of this section ordered SAMFs will be excluded, i.e. the assumption

$$c_1 < 0 < c_r \tag{3.10}$$

is in force, where $c_i := \log\lambda_i - \log\nu_i$. Still, whenever (3.10) is used as a necessary condition, it will be mentioned.
The complexity of the investigation of unordered SAMFs is founded in the fact that $J_\delta{}^+$ and $J_\delta{}^-$ are tight but not secure. First a relation to the somewhat simpler sets $I_n$ is established.

**Lemma 3.11** *Let $\delta_n = \lambda^n$, $k_\delta = \lceil \log\delta/\log\nu \rceil$ and $m_0 = \lceil 2\log\nu/\log\lambda \rceil$. Then*

$$J_\delta \subset \bigcup_{n=k_\delta}^{m_0 k_\delta} I_n$$
$$I_M \subset \bigcup_{n=M}^{m_0 M} J_{\delta_n}$$

*for all $\delta > 0$ and all integers $M$.*

**Proof**

**i)** Let $\delta > 0$ and take $\underline{i}$ from $J_\delta$. Set $n = |\underline{i}|$. Then

$$\nu^n \le \kappa(\underline{i}) \le \delta < \kappa(i_1 \dots i_{n-1}) \le \lambda^{n-1}$$

and hence $n \ge k_\delta$ and $n \le (\log\delta/\log\nu)(\log\nu/\log\lambda) + 1 \le m_0 k_\delta$. This proves the first inclusion.



**ii)** Let $M \in \mathbb{N}$ and take $\underline{i}$ with $|\underline{i}| = M$. There is a unique $n$ with $\delta_n < \kappa(i_1 \dots i_{M-1}) \le \delta_{n-1}$. From this $\kappa(\underline{i}) \le \lambda \cdot \kappa(i_1 \dots i_{M-1}) \le \delta_n$ and consequently $\underline{i}$ is an element of $J_{\delta_n}$. Moreover,

$$\nu^M \le \kappa(\underline{i}) \le \lambda^n = \delta_n < \kappa(i_1 \dots i_{M-1}) \le \lambda^{M-1},$$

thus $n > M-1$ and $n \le (\log\nu/\log\lambda)M \le m_0 M$. $\diamond$

As a consequence of lemma 3.11 we consider first the function

$$\sigma_n(a,b,\gamma) := \sigma(q,a,b,\gamma,I_n) = \underbrace{\sum_{\underline{i}\in I_n^+} p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^a \nu_{\underline{i}}{}^{\gamma-a}}_{\sigma_n^+(a,\gamma)} + \underbrace{\sum_{\underline{i}\in I_n^-} p_{\underline{i}}{}^q \nu_{\underline{i}}{}^b \lambda_{\underline{i}}{}^{\gamma-b}}_{\sigma_n^-(b,\gamma)} \tag{3.11}$$

where $q$ is regarded as a constant rather than as a variable. Since the asymptotic behaviour of $\sigma_n(a,b,\gamma)$ in $n$ is of interest, the two terms $\sigma_n^+$ and $\sigma_n^-$ have to be studied separately. They are both positive and strictly decreasing in $\gamma$. We will concentrate our study on $\sigma_n^+$. But note that $\sigma_n^-$ has exactly the same 'looks' with only one difference: the words $\underline{i}$ with $\lambda_{\underline{i}} = \nu_{\underline{i}}$ contribute to $\sigma_n^-$, not to $\sigma_n^+$. However, the proofs will be formulated in a manner to be correct also if it were just the other way round. So they will—mutatis mutandis—also be valid for $\sigma_n^-$.
As lemma 3.10 might suggest, the asymptotic behaviour of $\sigma_n^+$ is governed by the properties of the following 'characteristic function':

$$\chi(a,\gamma) := \sum_{i=1}^r p_i{}^q \lambda_i{}^a \nu_i{}^{\gamma-a}. \tag{3.12}$$

In order to apply limit theorems from probability theory consider the probability space $(I_\infty, \mathcal{B}, P^+)$ where $I_\infty$ is endowed with the product topology, $\mathcal{B}$ is the $\sigma$-algebra of its Borel sets and $P^+$ is the product measure on $\mathcal{B}$ induced by the measures

$$\{i\} \mapsto \frac{p_i{}^q \lambda_i{}^a \nu_i{}^{\gamma-a}}{\chi(a,\gamma)}$$

on the factors $\{1,\dots,r\}$ of $I_\infty$. Note that $P^+$ depends on $a$ and $\gamma$. Consequently they must be kept fixed when applying theorems from probability theory. The random variables

$$X_n : I_\infty \to \mathbb{R} \quad \underline{i}_\infty \mapsto c_{i_n}$$

are independent (by the property of the product measure) and identically distributed, i.e.

$$P^+[X_n = x] = \frac{1}{\chi(a,\gamma)} \sum_{i:\, c_i = x} p_i{}^q \lambda_i{}^a \nu_i{}^{\gamma-a}.$$

Their common expectation amounts

$$E[X_n] = \frac{1}{\chi(a,\gamma)} \sum_{i=1}^r c_i p_i{}^q \lambda_i{}^a \nu_i{}^{\gamma-a} = \frac{\chi_{.1}(a,\gamma)}{\chi(a,\gamma)}. \tag{3.13}$$

Here the partial derivate of $\chi$ with respect to the $k$-th variable is denoted by $\chi_{.k}$. The connection between $X_n$ and $\sigma_n^+$ is provided by the random variable

$$Z_n := \sum_{k=1}^{n} X_k$$

through the following lemma:

**Lemma 3.12** *For fixed $a$ and $\gamma$*

$$\sigma_n^+(a,\gamma) = \sum_{\underline{i}\in I_n^+} p_{\underline{i}}{}^q \lambda_{\underline{i}}^a \nu_{\underline{i}}^{\gamma-a} = \chi^n(a,\gamma)\cdot P^+[Z_n > 0].$$

**Proof** Let $c_1',\ldots,c_t'$ denote the $t$ different values of $c_1,\ldots,c_r$. For short, denote by $\sum'$ the sum over all $\underline{j} = j_1\ldots j_n \in \{1,\ldots,t\}^n$ such that $c_{j_1}'+\ldots+c_{j_n}' > 0$. Then

$$\begin{aligned} P^+[Z_n>0] &= {\sum}' P^+[X_1 = c_{j_1}',\ldots,X_n = c_{j_n}'] \\ &= {\sum}' P^+\Big[\{\underline{i}_\infty \in I_\infty : c_{i_k} = c_{j_k}'\ (k=1,\ldots,n)\}\Big] \\ &= P^+\Big[\{\underline{i}_\infty \in I_\infty : c_{i_1}+\ldots+c_{i_n} > 0\}\Big] = \chi^{-n}(a,\gamma)\sigma_n^+(a,\gamma). \end{aligned}$$

◇

To do the same with $\sigma_n^-$ just interchange $\lambda_i$ and $\nu_i$. The characteristic function is

$$\psi(b,\gamma) := \sum_{i=1}^{r} p_i{}^q \nu_i{}^b \lambda_i{}^{\gamma-b}. \tag{3.14}$$

The corresponding random variables are given through

$$P^-[Y_n = x] = \frac{1}{\psi(b,\gamma)} \sum_{i:\,d_i=x} p_i{}^q \nu_i{}^b \lambda_i{}^{\gamma-b},$$

where $d_i = \log\nu_i/\log\lambda_i = -c_i$, and

$$\sigma_n^-(b,\gamma) = \sum_{\underline{i}\in I_n^-} p_{\underline{i}}^q \nu_{\underline{i}}^b \lambda_{\underline{i}}^{\gamma-b} = \psi^n(b,\gamma)\cdot P^-[Y_1+\ldots+Y_n \ge 0] \tag{3.15}$$

for fixed $b$ and $\gamma$.

Turning back to $\sigma_n^+$ we keep $a$ and $\gamma$ fixed and consider first the case $E[X_n]\ge 0$. The Central Limit Theorem then tells us that $P^+[Z_n>0]$ is bounded away from zero, i.e. asymptotically greater than $1/2$. Consequently $\sigma_n^+$ roughly scales exponentially with base $\chi(a,\gamma)$. Defining $\gamma^+ = \gamma^+(a)$ to be the unique real number satisfying

$$\chi(a,\gamma^+) = \sum_{i=1}^{r} p_i{}^q \lambda_i{}^a \nu_i{}^{\gamma^+-a} = 1 \tag{3.16}$$



it is immediate that $\sigma_n^+(a,\gamma)$ is bounded if $\gamma=\gamma^+$ and tends exponentially to 0 if $\gamma>\gamma^+$ resp. to $\infty$ if $\gamma<\gamma^+$. Here the strict monotonicity of $\chi(a,\gamma)$ in $\gamma$ was used. Now assume that $E[X_n]<0$, i.e. $\chi_{.1}(a,\gamma)<0$. Then $P^+[Z_n>0]$ tends to zero exponentially with some base $\omega$ which is explicitly given by Chernoff's theorem [Bill, p 147]. Of course $\omega$ depends on $\gamma$ and on $a$ and so the question whether $\omega\cdot\chi(a,\gamma)$ is greater, equal or less than 1 arises. In order to give the value of $\omega$ the moment generating function of the random variable $X_n$ is required.

$$\begin{aligned} M(t) := E[e^{tX_n}] &= \sum_x e^{tx}\cdot P^+[X_n = x] = (\chi(a,\gamma))^{-1}\sum_{i=1}^{r} (\lambda_i/\nu_i)^t p_i{}^q \lambda_i{}^a \nu_i{}^{\gamma-a} \\ &= \frac{\chi(a+t,\gamma)}{\chi(a,\gamma)} = (\chi(a,\gamma))^{-1}\sum_{i=1}^{r} p_i{}^q \nu_i{}^\gamma e^{c_i(t+a)}. \end{aligned}$$

$M(t)$ is a strictly convex function of $t$ with a unique minimum $t^+$, due to (3.10). This $t^+ = t^+(a,\gamma)$ is determined by $M'(t^+)=0$, or equivalently

$$\chi_{.1}(a+t^+,\gamma) = \sum_{i=1}^{r} c_i p_i{}^q \nu_i{}^\gamma e^{c_i(t^++a)} = 0. \tag{3.17}$$

By Chernoff's theorem $\omega$ is just the minimal value of $M$, thus $M(t^+)$. The following lemma answers the question concerning $\omega\cdot\chi$.

**Lemma 3.13** *Given $a$ and provided $c_1<0<c_r$, the function*

$$h(\gamma) := \chi(a,\gamma)M(t^+(a,\gamma)) = \chi(a+t^+(a,\gamma),\gamma)$$

*is strictly decreasing, and there is a unique $\gamma_0$ such that $h(\gamma_0)=1$. In particular the equation system*

$$\left|\begin{array}{ccccc} \chi(a+t,\gamma) & = & \sum_{i=1}^{r} p_i{}^q\nu_i^\gamma e^{c_i(t+a)} & = & 1 \\ \chi_{.1}(a+t,\gamma) & = & \sum_{i=1}^{r} c_i p_i{}^q\nu_i^\gamma e^{c_i(t+a)} & = & 0 \end{array}\right| \tag{3.18}$$

*in the variables $\gamma$ and $t$ has a unique solution which is $(\gamma_0, t^+(a,\gamma_0))$. Moreover, $\gamma_0$ does not depend on $a$ and satisfies the inequality $\gamma_0\le\gamma^+(a)$.*

**Proof** Due to (3.10) $t^+$ is uniquely determined by the second equation of (3.18), which is actually (3.17). Moreover, $t^+$ depends continuously differentiable on $\gamma$ since $\chi_{.11}\neq 0$. The monotonicity of $h$ follows then readily:

$$h'(\gamma) = \chi_{.1}\cdot\tfrac{\partial}{\partial\gamma}t^+ + \chi_{.2} = \sum_{i=1}^{r}\log\nu_i\cdot p_i{}^q\nu_i{}^\gamma e^{c_i(a+t^+)} \le \log\lambda\cdot h(\gamma) < 0,$$

since $\chi_{.1}$ vanishes by definition of $t^+$. An application of the mean value theorem as in theorem 2.6 ii) shows that $h(\gamma)\to\infty$ $(\gamma\to-\infty)$. On the other hand,

$$h(\gamma) = \sum_{i=1}^{r} p_i{}^q\nu_i^\gamma e^{c_i(t^++a)} \le \sum_{i=1}^{r} p_i{}^q\nu_i^\gamma \to 0 \quad (\gamma\to\infty),$$

because $t^+$ minimizes by its definition the strictly convex function $t \mapsto \sum p_i{}^q \nu_i^\gamma e^{c_i(t+a)}$ for fixed $a$ and $\gamma$. This establishes the existence and the uniqueness of $\gamma_0$ and hence the solvability of (3.18).
Regarding the equation system, the independence of $\gamma_0$ from $a$ is immediate. Finally note that $M(t^+) \leq M(0) = 1$ for all $\gamma$. Hence $h(\gamma^+(a)) \leq \chi(a, \gamma^+(a)) = 1 = h(\gamma_0)$ and $\gamma_0 \leq \gamma^+(a)$. $\diamond$

**Remark** As the close relation to the proof of theorem 2.6 may suggest, it is indeed possible to calculate the spectrum $F$ of a SMF directly, i.e. not as the Legendre transform of $T$ but rather by using the methods introduced in this section. Without going into details we sketch the interesting fact, how the Legendre relation between $F$ and $T$ is hidden in Chernoff's theorem. The $\sum \lambda_{\underline{i}}{}^\gamma$ over all words $\underline{i} \in J_\delta$ satisfying $p_{\underline{i}} \geq \lambda_{\underline{i}}{}^\alpha$ approximates $N_\delta(\alpha) \cdot \delta^\gamma$, and hence determines $F^+(\alpha)$ as the one $\gamma$ for which this sum is asymptotically bounded. As lemma 3.11 suggests it is enough to consider the sum over all $\underline{i} \in I_n$ with the same property. The random variables $\overline{X}_n$, which attain the values $\log(p_i/\lambda_i^\alpha)$ with probability $\lambda_i^\gamma / \sum \lambda_i^\gamma$, provide the connection to probability theory. A sophisticated study reveals that Chernoff's theorem applies exactly for $\alpha < \alpha_0$, leading to

$$\frac{1}{n} \log \sum_{\substack{|\underline{i}|=n \\ p_{\underline{i}} \geq \lambda_{\underline{i}}{}^\alpha}} \lambda_{\underline{i}}{}^\gamma = \frac{1}{n} \log \big(\sum_{i=1}^r \lambda_i^\gamma\big)^n \overline{P}[\overline{X}_1 + \ldots + \overline{X}_n \geq 0] \to \log \Big(\sum_{i=1}^r \lambda_i^\gamma \cdot \inf_t M(t)\Big).$$

Thereby $M(t) = E[e^{t\overline{X}_n}] = (\sum \lambda_i^\gamma)^{-1} \sum \lambda_i^\gamma (p_i/\lambda_i^\alpha)^t$. Consequently, $F^+(\alpha)$ is the one $\gamma$ for which $\inf_t \sum \lambda_i^\gamma (p_i/\lambda_i^\alpha)^t = 1$. This is exactly what (2.23) expresses. On the other hand, these equations just say that $F^+$ is the Legendre transform of $T$. In the case $\lambda_i = \lambda$ it reads most explicitly as

$$\inf_t \Big(\alpha t - \gamma + \frac{\log \sum_{i=1}^r p_i^t}{\log 1/\lambda}\Big) = -\gamma + \inf_t \Big(\alpha t + T(t)\Big) = 0.$$

$\diamond$

We are now in the position to give the asymptotic behaviour of $\sigma_n^+$.

**Lemma 3.14** *Assume $c_1 < 0 < c_r$ and define*

$$\Gamma^+(a) := \begin{cases} \gamma^+(a) & \textit{if } \chi_{.1}(a, \gamma^+(a)) \geq 0 \\ \gamma_0 & \textit{otherwise.} \end{cases}$$

*Then*

$$\lim_{n\to\infty} \frac{1}{n} \log \sigma_n^+(a, \gamma) \begin{cases} < 0 & \textit{if } \gamma > \Gamma^+(a), \\ = 0 & \textit{if } \gamma = \Gamma^+(a), \\ > 0 & \textit{if } \gamma < \Gamma^+(a). \end{cases}$$

**Proof** Fix $a \in \mathbb{R}$.



**i)** First, $\gamma$ shall be fixed too. In order to obtain the asymptotics of $\sigma_n^+$ it is enough to know the one of $P^+[Z_n > 0]$. Assume first that the expectation $E[X_n] \geq 0$, i.e. $\chi_{.1}(a, \gamma) \geq 0$. The common variance of the $X_n$ is var $= E[(X_n - E[X_n])^2]$ and vanishes exactly if $c_1 = \ldots = c_r = E[X_n]$, a case which is excluded. Thus, the Central Limit Theorem tells us that

$$1 \geq P^+[Z_n \geq 0] \geq P^+[Z_n > 0] \geq P^+[Z_n > nE] = P^+[\frac{Z_n - nE}{n\sqrt{\text{var}}} > 0] \to \frac{1}{2}.$$

Assume now that $E[X_n] < 0$. Since $P^+[X_n > 0] \geq p_r > 0$ by (3.10), Chernoff's theorem [Bill, p 147] implies:

$$\lim_{n\to\infty} \frac{1}{n} \log P^+[Z_n \geq 0] = \lim_{n\to\infty} \frac{1}{n} \log P^+[Z_n > 0] = \log M(t^+(a, \gamma)).$$

Summarizing

$$\lim_{n\to\infty} \frac{1}{n} \log \sigma_n^+(a, \gamma) = \begin{cases} \log \chi(a, \gamma) & \text{if } \chi_{.1}(a, \gamma) \geq 0, \\ \log h(\gamma) & \text{otherwise.} \end{cases}$$

**ii)** From now on, $\gamma$ is variable in $\mathbb{R}$. The base of the exponential growth of $\sigma_n^+$ is switching between two functions of $\gamma$ according to i). However, it is important to recognize that such a change is impossible in the $\gamma$ interval $[\gamma_0, \gamma^+(a)]$: Assume $\chi_{.1}(a, \gamma) = 0$ for some $\gamma \in [\gamma_0, \gamma^+(a)]$. Then, $t^+(a, \gamma) = 0$ and $h(\gamma) = \chi(a, \gamma)$. But since $h$ and $\chi$ are both strictly monotonous decreasing, and since $h(\gamma_0) = 1 = \chi(a, \gamma^+(a))$, this implies $\gamma^+(a) = \gamma_0$. Note in addition the equivalence of the following four conditions, which is an immediate consequence of (3.18):

$$\chi_{.1}(a, \gamma^+(a)) = 0 \quad \chi_{.1}(a, \gamma_0) = 0 \quad t^+(a, \gamma^+(a)) = 0 \quad \gamma^+(a) = \gamma_0. \tag{3.19}$$

**iii)** From $M(0) = 1$ follows $h(\gamma) \leq \chi(a, \gamma)$. Thus, $\chi(a, \gamma)$ and $h(\gamma)$ are both strictly greater than 1 for $\gamma < \gamma_0$ resp. strictly less than 1 for $\gamma > \gamma^+(a)$. For these $\gamma$ it does not matter which case of i) applies. For the remaining $\gamma$ ii) yields: Either $\chi_{.1}(a, \gamma^+) \geq 0$ for all $\gamma \in [\gamma_0, \gamma^+(a)]$, $\Gamma^+(a) = \gamma^+(a)$ and the investigated limes is $\log \chi(a, \gamma)$; or $\chi_{.1}(a, \gamma^+) < 0$ for all $\gamma \in [\gamma_0, \gamma^+(a)]$, $\Gamma^+(a) = \gamma_0$ and the limes is $\log h(\gamma)$. This proves the claim. $\diamond$

Turning again to $\sigma_n^-$ define $\gamma^- = \gamma^-(b)$ to be the unique real number satisfying

$$\psi(b, \gamma^-) = \sum_{i=1}^r p_i{}^q \nu_i{}^b \lambda_i{}^{\gamma^- - b} = 1. \tag{3.20}$$

Next, consider the equation system

$$\left| \begin{array}{rcl} \psi(b+t, \gamma) & = & \sum_{i=1}^r p_i{}^q \lambda_i^\gamma e^{d_i(t+b)} = 1 \\ \psi_{.1}(b+t, \gamma) & = & \sum_{i=1}^r d_i p_i{}^q \lambda_i^\gamma e^{d_i(t+b)} = 0 \end{array} \right|$$

Interchanging $\lambda_i$ with $\nu_i$ the existence of a unique solution can be deduced from lemma 3.13. Moreover, $\psi(b,\gamma) = \chi(\gamma - b, \gamma)$ for all $b$ and $\gamma$ and this solution must be $\gamma = \gamma_0$, $t = -t^+(\gamma_0 - b, \gamma_0)$.
Setting

$$\Gamma^-(a) := \begin{cases} \gamma^-(b) & \text{if } \psi_{.1}(b,\gamma^-(b)) \geq 0 \\ \gamma_0 & \text{otherwise,} \end{cases}$$

with $\gamma_0$ from lemma 3.13, one finds

$$\lim_{n\to\infty} \frac{1}{n} \log \sigma_n^-(b,\gamma) \begin{cases} < 0 & \text{if } \gamma > \Gamma^-(b), \\ = 0 & \text{if } \gamma = \Gamma^-(b), \\ > 0 & \text{if } \gamma < \Gamma^-(b). \end{cases} \tag{3.21}$$

from (3.15) and the proof of lemma 3.14.
To get an illuminating picture note that the three relevant values $\gamma^+(a)$, $\gamma^-(b)$ and $\gamma_0$ are found on the level curve $\chi \equiv 1$ in the $(a,\gamma)$-plane (see Fig. 3.6). The curve

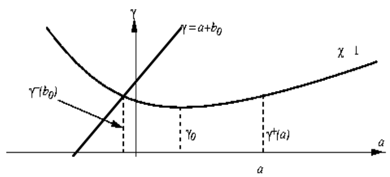


Figure 3.6: *The features of a typical level curve $\chi \equiv 1$ where $c_1 < 0 < c_r$. The curve equals the graph of the function $a \mapsto \gamma^+(a)$. The picture illustrates the slow convergence of numerical methods bound to find $\gamma_0$.*

is the graph of the function $a \mapsto \gamma^+(a)$. Moreover, it intersects every straight line $a \mapsto \gamma = a + b_0$ exactly once in the point $(\gamma^-(b_0) - b_0, \gamma^-(b_0))$. Finally, it has a unique minimum due to $c_1 < 0 < c_r$ and the corresponding minimal value is just $\gamma_0$. This illustrates the independence of $\gamma_0$ from $a$ once more.
Now that the behaviour of $\sigma_n(a,b,\gamma)$ is known, the conclusions for the sums over the somewhat more complicated sets $J_\delta$ may be drawn.

**Proposition 3.15** *Assume $c_1 < 0 < c_r$ and let $\delta_n = \lambda^n$. Then*

$$\begin{aligned} \lim_{\delta\downarrow 0} \ \sigma(q,a,b,\gamma,J_\delta) &= 0 \quad \textit{if } \gamma > \max(\Gamma^+(a),\Gamma^-(b)), \\ \limsup_{n\to\infty} \ \sigma(q,a,b,\gamma,J_{\delta_n}) &= \infty \quad \textit{if } \gamma < \max(\Gamma^+(a),\Gamma^-(b)). \end{aligned}$$

**Proof**

**i)** Take first $\gamma > \max(\Gamma^+(a),\Gamma^-(b))$. Lemma 3.14 and (3.21) give $L_1 < 1$ and $L_2 < 1$ such that $\sigma_n(a,b,\gamma) = \sigma_n^+(a,\gamma) + \sigma_n^-(b,\gamma) \leq L_1^n + L_2^n$ for $n$ large



enough. Lemma 3.11 implies for sufficiently small $\delta > 0$

$$\sigma(q,a,b,\gamma,J_\delta) \leq \sum_{n=k_\delta}^{m_0k_\delta} \sigma(q,a,b,\gamma,I_n) \leq \sum_{n=k_\delta}^{m_0k_\delta} {L_1}^n + {L_2}^n \leq m_0 k_\delta({L_1}^{k_\delta} + {L_2}^{k_\delta}).$$

The first part of the theorem then follows from $k_\delta \to \infty$ $(\delta \to 0)$.

**ii)** Take now $\gamma < \max(\Gamma^+(a),\Gamma^-(b))$. Since $\sigma_n^+$ and $\sigma_n^-$ are both positive, (3.21) and lemma 3.14 provide a number $L > 1$ such that $\sigma_n(a,b,\gamma) \geq L^n$ for sufficiently large $n$. Then

$$\sum_{n=M}^{m_0M} \sigma(q,a,b,\gamma,J_{\delta_n}) \geq \sigma_M(a,b,\gamma) \geq L^M$$

for large $M$. Since the terms on the left hand side are all positive there must be an integer $n(M)$ between $M$ and $m_0M$ with

$$\sigma(q,a,b,\gamma,J_{\delta_{n(M)}}) \geq \frac{1}{m_0M} L^M.$$

This completes the proof. ◇

## 3.3 Generalized Dimensions

In this section the geometric properties of SAMFs and the limit proposition 3.15 are fused in order to estimate the singularity exponents $T(q)$. The lower and the upper bound differ in general, but they coincide provided that $T^{(1)}(q)$ and $T^{(2)}(q)$ are grid-regular. Moreover, as shall be shown, the latter is a sufficient condition for the grid-regularity of $T$. Finally the differentiability of $T$ will be investigated.

### 3.3.1 Estimate

The notation of section 3.2 has to be adapted. Now the dependence on $q$ has to be indicated explicitly. On the other hand, the variables $a$ and $b$ will only take some particular values. Moreover, a formula should be provided which includes ordered SAMFs. To keep the following definitions at reasonable size set $a = T^{(1)}(q)$, $b = T^{(2)}(q)$. Denote the unique solution (in $\gamma$) of

$$\sum_{i=1}^r {p_i}^q {\lambda_i}^a {\nu_i}^{\gamma - a} = 1 \quad \text{resp.} \quad \sum_{i=1}^r {p_i}^q {\nu_i}^b {\lambda_i}^{\gamma - b} = 1$$

by $\gamma^+(q)$, resp. $\gamma^-(q)$. If $\lambda_i < \nu_i$ and $\lambda_j > \nu_j$ for some $i$ and some $j$, denote the unique solution (in $(x,\gamma)$) of

$$\left| \begin{aligned} \sum_{i=1}^r {p_i}^q {\lambda_i}^x {\nu_i}^{\gamma - x} &= 1 \\ \sum_{i=1}^r \log(\lambda_i/\nu_i) {p_i}^q {\lambda_i}^x {\nu_i}^{\gamma - x} &= 0 \end{aligned} \right|$$

by $(a_0(q), \gamma_0(q))$, otherwise set $\gamma_0(q) = -\infty$. Finally set

$$\Gamma^+(q) := \begin{cases} \gamma^+(q) & \text{if } \sum_{i=1}^r \log(\lambda_i/\nu_i) {p_i}^q {\lambda_i}^a {\nu_i}^{\gamma^+(q)-a} \geq 0, \\ \gamma_0(q) & \text{otherwise,} \end{cases}$$

$$\Gamma^-(q) := \begin{cases} \gamma^-(q) & \text{if } \sum_{i=1}^r \log(\nu_i/\lambda_i) {p_i}^q {\nu_i}^b {\lambda_i}^{\gamma^-(q)-b} \geq 0, \\ \gamma_0(q) & \text{otherwise,} \end{cases}$$

and define

$$\boxed{\Gamma(q) := \max\left(\Gamma^+(q), \Gamma^-(q)\right).} \tag{3.22}$$

In order to provide lower bounds, the analog definitions have to be carried out with $a = \underline{T}^{(1)}(q)$ and $b = \underline{T}^{(2)}(q)$. The corresponding functions will be denoted by $\underline{\gamma}^+(q)$, $\underline{\gamma}^-(q)$, $\underline{\gamma}_0(q)$, $\underline{\Gamma}^+(q)$, $\underline{\Gamma}^-(q)$ and $\underline{\Gamma}(q)$.

**Proposition 3.16** *Let $\mu$ be a SAMF and let $q \geq 0$. Then*

$$\underline{\Gamma}(q) \leq T(q) \leq \Gamma(q).$$

*The upper bound is also valid for negative $q$, and, provided $\mu$ is centered, the lower bound as well. Moreover, for ordered C-SAMFs the left hand side even bounds $\underline{T}(q)$ from below.*

**Proof** Let $q \in \mathbb{R}$.

**i)** Take $\gamma > \Gamma(q)$ arbitrarily and choose $\eta > 0$ such that $\gamma - \eta > \Gamma$. Lemma 3.5 (if $q \geq 0$) resp. lemma 3.6 (if $q < 0$) and proposition 3.15 say, that

$$S_\delta(q)\delta^\gamma \leq \text{const} \cdot \sigma(q, T^{(1)}(q), T^{(2)}(q), \gamma - \eta, J_\delta) \leq 1$$

for all sufficiently small $\delta > 0$. This gives the upper bound.

**ii)** Take $\gamma < \underline{\Gamma}(q)$ arbitrarily and choose $\eta > 0$ such that $\gamma + \eta < \underline{\Gamma}$. Lemma 3.5 (if $q \geq 0$) resp. lemma 3.9 (if $q < 0$), which only applies to C-SAMFs, implies

$$S_\delta(q)\delta^\gamma \geq \text{const} \cdot \sigma(q, \underline{T}^{(1)}(q), \underline{T}^{(2)}(q), \gamma + \eta, J_{\delta'})$$

for arbitrary $\delta > 0$ and for a particular multiple $\delta'$ of $\delta$. Since $c \cdot \lambda^n$ is an admissible sequence, proposition 3.15 gives $T(q) \geq \gamma$, but no information about $\underline{T}(q)$.

**iii)** Assume now $\lambda_i \geq \nu_i$ $(i = 1, \ldots, r)$. Note first that $\underline{\gamma}^+(q) = \underline{\Gamma}(q)$: either $\lambda_i = \nu_i$ for $i = 1, \ldots, r$ and the solutions $\underline{\gamma}^+ = \underline{\gamma}^- = \underline{\gamma}_0$ coincide, or $\underline{\Gamma}^+ \equiv \underline{\gamma}^+$ and $\underline{\Gamma}^- \equiv \underline{\gamma}_0$. Take now $\eta > 0$ arbitrarily. Provided $\mu$ is a C-SAMF lemma 3.9 and lemma 3.10 yield

$$S_\delta(q)\delta^{\gamma-\eta} \geq c_4 \sigma(q, a, b, \gamma, J_{\delta'}) = c_4$$

for all $\delta > 0$. Hence $\underline{T}(q) \geq \gamma - \eta$ and the lower bound is also valid for $\underline{T}(q)$. A similar argument applies to the case $\lambda_i \leq \nu_i$ $(i = 1, \ldots, r)$. $\diamond$



### 3.3.2 Grid-Regularity

Of course, the value of $T(q)$ is determined by the proposition above as soon as $\underline{T}^{(1)}(q) = T^{(1)}(q)$ and $\underline{T}^{(2)}(q) = T^{(2)}(q)$ are given. In the case of ordered SAMFs less assumptions give even the grid-regularity of $T(q)$:

**Corollary 3.2** *Let $\mu$ be a SAMF with $\lambda_i \geq \nu_i$ $(i = 1, \ldots, r)$ and let $q$ be a real number for which $T^{(1)}(q)$ is grid-regular. If $q < 0$ assume in addition that $\mu$ is vertically centered. Then $T(q)$ is grid-regular too and*

$$\sum_{i=1}^r {p_i}^q (\lambda_i/\nu_i)^{T^{(1)}(q)} {\nu_i}^{T(q)} = 1.$$

For $\lambda_i = \nu_i$ $(i = 1, \ldots, r)$, the formula for $T(q)$ reduces to the earlier equation (2.22)

$$\sum_{i=1}^r p_i^q \lambda_i^{T(q)} = 1.$$

**Proof** Lemma 3.9 holds although $\mu$ is only vertically centered. This is obvious since ${J_\delta}^+ = J_\delta$. Moreover, $\underline{\gamma}^+(q) = \underline{\Gamma}(q) = \Gamma(q) = \gamma^+(q)$ and proposition 3.16 proves the claim. $\diamond$

Also the singularity exponents of an arbitrary SAMF are grid-regular, provided $T^{(1)}(q)$ and $T^{(2)}(q)$ are. Though this may not be surprising, the proof needs a new idea.

**Theorem 3.3** *Let $\mu$ be an arbitrary SAMF. If $q \geq 0$ is such that $T^{(1)}(q)$ and $T^{(2)}(q)$ are grid-regular, then $T(q)$ is grid-regular too and*

$$\boxed{T(q) = \Gamma(q)}.$$

*The assertion holds also for negative $q$ provided the measure is centered.*

**Remark** The condition $\underline{T}^{(k)}(q) = T^{(k)}(q)$ is certainly satisfied for SAMFs with self-similar projections.

**Proof** Take $q$ as in the statement and set $a = T^{(1)}(q)$, $b = T^{(2)}(q)$ for short. The equality $T = \Gamma$ is immediate. To prove the grid-regularity some preparation is needed.

**i)** First, let us prove a kind of monotonicity of $\sigma$: if $\sigma(q, a, b, \gamma, I_1) \leq 1$ and $\gamma \leq a+b$, then

$$\sigma(q, a, b, \gamma, J_\delta) \leq 1 \tag{3.23}$$

for all $\delta > 0$. Recall the recursive construction of $J_\delta$ (2.4) on page 34. The proof is by induction on this construction. For $J(1) = I_1$ (3.23) is trivial. Assuming $\sigma(q, a, b, \gamma, J(m)) \leq 1$ the same will be verified for $J := J(m + 1)$. It is enough to know

$$\sigma(q, a, b, \gamma, J(m)) \cdot \sigma(q, a, b, \gamma, I_1) \geq \sigma(q, a, b, \gamma, J),$$

and therefore enough to know, that

$$\lambda_{\underline{i}}{}^{a}\nu_{\underline{i}}{}^{\gamma-a}\cdot\sigma(q,a,b,\gamma,I_1)\geq\sum_{\underline{i}*k\in J^+}\lambda_{\underline{i}*k}{}^{a}\nu_{\underline{i}*k}{}^{\gamma-a}+\sum_{\underline{i}*k\in J^-}\nu_{\underline{i}*k}{}^{b}\lambda_{\underline{i}*k}{}^{\gamma-b},$$

for any $\underline{i}\in J(m)^+$ and mutatis mutandis for $\underline{i}\in J(m)^-$. Set $\underline{j}=\underline{i}*k$ for short and pass through the possible cases:

$$\begin{array}{lllll}
\lambda_{\underline{i}}>\nu_{\underline{i}}, & \lambda_k>\nu_k & : & \underline{j}\in J^+, & \lambda_{\underline{j}}{}^{a}\nu_{\underline{j}}{}^{\gamma-a} = (\lambda_{\underline{i}}{}^{a}\nu_{\underline{i}}{}^{\gamma-a})(\lambda_k{}^{a}\nu_k{}^{\gamma-a}) \\
 & \lambda_k\leq\nu_k, & \lambda_{\underline{j}}>\nu_{\underline{j}}: & \underline{j}\in J^+, & \lambda_{\underline{j}}{}^{a}\nu_{\underline{j}}{}^{\gamma-a} \leq (\lambda_{\underline{i}}{}^{a}\nu_{\underline{i}}{}^{\gamma-a})(\nu_k{}^{b}\lambda_k{}^{\gamma-b}) \\
 & & \lambda_{\underline{j}}\leq\nu_{\underline{j}}: & \underline{j}\in J^-, & \nu_{\underline{j}}{}^{b}\lambda_{\underline{j}}{}^{\gamma-b} \leq (\lambda_{\underline{i}}{}^{a}\nu_{\underline{i}}{}^{\gamma-a})(\nu_k{}^{b}\lambda_k{}^{\gamma-b})
\end{array}$$

Here $y^b x^{\gamma-b}\geq y^{\gamma-a}x^a$ for $y\geq x$ was used, and (3.23) is proven. Similarly

$$\sigma(q,a,b,\gamma,J_\delta)\geq 1$$

for all $\delta>0$ provided $\sigma(q,a,b,\gamma,I_1)\geq 1$ and $\gamma\geq a+b$.

**ii)** To get an intuition note that any $\gamma$ satisfying (3.23) must by proposition 3.15 resp. lemma 3.10 be greater or equal to $\Gamma(q)$. However, it will even be proven $\Gamma(q)=\lim\gamma(J_\delta)$, where $\gamma(J_\delta)$ denotes the unique solution of

$$\sigma(q,a,b,\gamma(J_\delta),J_\delta)=1.$$

This property of $\gamma(J_\delta)$ will render an estimate of $\sigma(q,a,b,\gamma,J_\delta)$ for all $\delta$.

**iii)** Fix $\delta>0$ for the moment and assume $\gamma(J_\delta)\leq a+b$. Certainly

$$\mu=\sum_{i=1}^{r}p_i w_{i*}\mu=\sum_{\underline{i}\in J_\delta}p_{\underline{i}}\cdot w_{\underline{i}*}\mu,$$

again by the construction of $J_\delta$. Let $s:=\#J_\delta$ and let $\underline{j}:\{1,\ldots,s\}\to J_\delta$, $l\mapsto\underline{j}(l)$ be an enumeration of $J_\delta$. Furthermore set $w_l^*:=w_{\underline{j}(l)}$ and $p_l^*:=p_{\underline{j}(l)}$, and denote the relevant values of the associated multifractal

$$\mu^*:=\langle w_1^*,\ldots,w_s^*;p_1^*,\ldots,p_s^*\rangle$$

by $\sigma^*$, $S_\varepsilon{}^*(q)$, $T^*$ and so on. In particular $I_1^*=\{1,\ldots,s\}$ corresponds to $J_\delta$ and $\sigma^*(q,a,b,\gamma,I_1^*)=\sigma(q,a,b,\gamma,J_\delta)$. On the other hand, $\mu^*=\mu$ by the above invariance, which means that only a coarser IFS was chosen for the same multifractal. Thus all geometrical values coincide, e.g. $S_\varepsilon{}^*(q)=S_\varepsilon(q)$, $T^*=T$, $T^{(k)^*}=T^{(k)}$ and so on. Now apply (3.23), lemma 3.5 and lemma 3.6 to the coarser IFS. By definition of $\gamma(J_\delta)$

$$\sigma^*(q,a,b,\gamma(J_\delta),J_\varepsilon{}^*)\leq 1$$



for all $\varepsilon>0$. Given $\eta>0$, $S_\varepsilon{}^*(q)\varepsilon^{\gamma(J_\delta)+\eta}$ is bounded from above independently of $\varepsilon>0$. Thus $T^*(q)\leq\gamma(J_\delta)+\eta$. Altogether

$$T(q)=T^*(q)\leq\gamma(J_\delta).$$

Mutatis mutandis the assumption $\gamma(J_\delta)\geq a+b$ leads with lemma 3.5 resp. lemma 3.9 to

$$\underline{T}(q)\geq\gamma(J_\delta).$$

**iv)** After all these preliminaries let us prove the assertion of the theorem. Take first the case $\gamma(I_1)\leq a+b$. Then

$$\sigma(q,a,b,\gamma(I_1),J_\delta)\leq 1$$

by (3.23) and, since $\sigma$ is decreasing in $\gamma$, $\gamma(J_\delta)\leq\gamma(I_1)\leq a+b$. Consequently, $T(q)\leq\gamma_{\inf}:=\inf_{\delta>0}(\gamma(J_\delta))$ by iii). On the other hand,

$$\sigma(q,a,b,\gamma_{\inf},J_\delta)\geq\sigma(q,a,b,\gamma(J_\delta),J_\delta)=1$$

for all $\delta>0$ and by lemma 3.5 resp. lemma 3.9 $\underline{T}(q)\geq\gamma_{\inf}$. This yields indeed

$$\underline{T}(q)=T(q)=\inf_{\delta>0}(\gamma(J_\delta)).$$

**v)** If $\gamma(I_1)\geq a+b$ then $\underline{T}(q)\geq\gamma_{\sup}:=\sup_{\delta>0}(\gamma(J_\delta))$ by iii). With lemma 3.5 resp. lemma 3.6 and

$$\sigma(q,a,b,\gamma_{\sup},J_\delta)\leq\sigma(q,a,b,\gamma(J_\delta),J_\delta)=1$$

for all $\delta>0$, which leads to $T(q)\leq\gamma_{\sup}$ and

$$\underline{T}(q)=T(q)=\sup_{\delta>0}(\gamma(J_\delta)).$$

◇

### 3.3.3 Differentiability

Let us turn to the question of differentiability. For ordered SAMFs the answer is readily given by corollary 3.2 and the implicit function theorem: the regularities of $T^{(1)}$ resp. $T^{(2)}$ carry over to $\gamma^+(q)$ resp. $\gamma^-(q)$ and hence to $T$ itself.
For general SAMFs $\gamma_0(q)$ has to be taken into account as well. Since $\gamma_0$ depends $C^1$ on $q$, the only difficulty is at values $q$ where $\Gamma$ switches from one of the three candidates $\gamma^+$, $\gamma^-$ and $\gamma_0$ to another. In this context we can prove at least:

**Proposition 3.17** *If $T^{(1)}(q)$ is differentiable in a neighbourhood of $q_0$, then $\Gamma^+(q)$ has the same property. Similar for $T^{(2)}$ and $\Gamma^-$.*

**Proof** Here, the dependence of $\chi$ (3.12) on $q$ has to be expressed explicitly:

$$\chi(a,\gamma,q) := \sum_{i=1}^{r} {p_i}^q {\lambda_i}^a {\nu_i}^{\gamma-a}.$$

For ordered SAMFs the assertion is obvious. Thus assume without loss of generality $\lambda_1 < \nu_1$ and $\lambda_r > \nu_r$.

**i)** The pair of equations $\chi(a,\gamma,q) = 1$, $\chi_{.1}(a,\gamma,q) = 0$ is uniquely solved by some $(a_0,\gamma_0)$ depending on $q$, where $a_0$ is actually not of interest here. Since

$$\det \begin{pmatrix} \chi_{.1} & \chi_{.2} \\ \chi_{.11} & \chi_{.12} \end{pmatrix} = -\chi_{.11}\chi_{.2} > 0$$

at $(a,\gamma) = (a_0,\gamma_0)$, the solutions $a_0$ and $\gamma_0$ depend differentiably on $q$. Taking the implicit derivate of the first equation and observing $\chi_{.1} = 0$ yields

$$\tfrac{d}{dq}\gamma_0(q) = -\frac{\chi_{.3}(a_0,\gamma_0,q)}{\chi_{.2}(a_0,\gamma_0,q)}. \tag{3.24}$$

In particular $\gamma_0 \in C^1$.

**ii)** From $\chi(T^{(1)}(q),\gamma^+(q),q) \equiv 1$ and $\chi_{.2} > 0$ follows

$$\tfrac{d}{dq}\gamma^+(q) = -\frac{\chi_{.3}(T^{(1)},\gamma^+,q) + \chi_{.1}(T^{(1)},\gamma^+,q)(T^{(1)})'(q)}{\chi_{.2}(T^{(1)},\gamma^+,q)} \tag{3.25}$$

So, $\gamma^+(q)$ is differentiable near $q_0$.

**iii)** Since $\chi_{.1}(T^{(1)},\gamma^+,q)$ is a continuous function of $q$ near $q_0$, a switch from $\gamma_0(q)$ to $\gamma^+(q)$ in the value of $\Gamma^+(q)$ is only possible at a zero $\overline{q}$ of $\chi_{.1}$. But then $\gamma_0(\overline{q}) = \gamma^+(\overline{q})$ by (3.19). Moreover, $a_0(\overline{q}) = T^{(1)}(\overline{q})$ is easily verified, and the derivates of $\gamma_0$ and $\gamma^+$ coincide by i) and ii).

**iv)** The argumentation for $\Gamma^-$ is similar. $\diamond$

This proposition has consequences:

- The differentiability of $T^{(1)}$ and $T^{(2)}$ carries over to $\Gamma = \max(\Gamma^+,\Gamma^-)$ except at points where the maximum causes a wedge. This may only happen when $\gamma^+(q) = \gamma^-(q)$.
- With the possibility of wedges a completely new feature appears, which is not encountered among the singularity exponents of SMFs. This may have consequences for the search of models.
- There are IFS, i.e. multiplicative cascades, producing nondifferentiable singularity exponents, not only carefully constructed examples.



The section closes with an astonishing and satisfying result: At $q = 1$ one would expect a wedge, since $\gamma^+(1) = \gamma^-(1) = 0$. Nevertheless $T$ is always differentiable at $q = 1$ and the interesting value $D_1$ can be given, provided that $T^{(1)}$ and $T^{(2)}$ are grid-regular and $C^1$ near 1.

**Corollary 3.4** *For any SAMF $\mu$ the following implications hold:*
*If $\sum_{i=1}^{r} p_i \log(\lambda_i/\nu_i) > 0$ and $T^{(1)}$ is grid-regular and $C^1$ near 1, then*

$$D_1 = \frac{\sum_{i=1}^{r} p_i(\log p_i - {D_1}^{(1)}\log(\lambda_i/\nu_i))}{\sum_{i=1}^{r} p_i \log \nu_i},$$

*if $\sum_{i=1}^{r} p_i \log(\lambda_i/\nu_i) < 0$ and $T^{(2)}$ is grid-regular and $C^1$ near 1, then*

$$D_1 = \frac{\sum_{i=1}^{r} p_i(\log p_i - {D_1}^{(2)}\log(\nu_i/\lambda_i))}{\sum_{i=1}^{r} p_i \log \lambda_i},$$

*finally, if $\sum_{i=1}^{r} p_i \log(\lambda_i/\nu_i) = 0$ and $T^{(1)}$ and $T^{(2)}$ are grid-regular and $C^1$ near 1, then*

$$D_1 = \frac{\sum_{i=1}^{r} p_i \log p_i}{\sum_{i=1}^{r} p_i \log \lambda_i} = \frac{\sum_{i=1}^{r} p_i \log p_i}{\sum_{i=1}^{r} p_i \log \nu_i}.$$

**Proof**

**i)** Certainly $T^{(1)}(1) = T^{(2)}(1) = 0$, thus $\gamma^+(1) = \gamma^-(1) = 0$. Consequently, the conditions in the definitions of $\Gamma^+$ and $\Gamma^-$ reduce to $\sum p_i \log(\lambda_i/\nu_i) \geq 0$ and $\sum p_i \log(\lambda_i/\nu_i) \leq 0$, respectively.

**ii)** Assume first that $\sum p_i \log(\lambda_i/\nu_i) > 0$ and that $T^{(1)}(q)$ is grid-regular and $C^1$ near 1. Then, for continuity reasons, $\underline{\Gamma}(q) = \underline{\gamma}^+(q) = \gamma^+(q) = \Gamma(q)$ near $q = 1$, since $\gamma_0 \leq \gamma^+$. With proposition 1.19 and (3.25)

$$D_1 = -T'(1) = -\tfrac{d}{dq}\gamma^+(q)\big|_{q=1} = \frac{\chi_{.3}(0,0,1) + \chi_{.1}(0,0,1)(T^{(1)})'(1)}{\chi_{.2}(0,0,1)}.$$

The case $\sum p_i \log(\lambda_i/\nu_i) < 0$ is obtained by interchanging $\lambda_i$ with $\nu_i$ and replacing $T^{(1)}(q)$ by $T^{(2)}(q)$.

**iii)** At last assume that $\sum p_i \log(\lambda_i/\nu_i) = 0$ and that $T^{(1)}(q)$ and $T^{(2)}(q)$ are grid-regular and $C^1$ near 1. Unless $\lambda_i = \nu_i$ $(i = 1,\ldots,r)$—which is a trivial case—$\lambda_1 < \nu_1$ and $\lambda_r > \nu_r$ can be assumed without loss of generality. Then $\gamma_0(1) = \gamma^+(1) = 0$ by (3.19), which leads with (3.24) and ii) to

$$\tfrac{d}{dq}\gamma_0(q)\big|_{q=1} = \tfrac{d}{dq}\gamma^+(q)\big|_{q=1} = \tfrac{d}{dq}\gamma^-(q)\big|_{q=1} = -\frac{\sum_{i=1}^r p_i \log p_i}{\sum_{i=1}^r p_i \log \lambda_i}.$$

So $\Gamma^+$ and $\Gamma^-$ have the same derivate and the same value at 1, and since they are continuously differentiable near 1 (Prop. 3.17) the mean value theorem of calculus shows that their maximum is differentiable at 1 as well. ◇

## 3.4 Box and Hausdorff Dimension

In this section simple formulas for the special value $D_0 = T(0) = d_{\text{box}}(K)$ and for the 'almost sure' Hausdorff dimension of $K$ are provided. While only the general case is treated in this section, most explicit results in a slightly less general case are spotted in subsection 3.5.2. To give some history and also for later use the inspiring results of Douady et Oesterlé [DO] and Falconer [Falc3] are stated first.
The main result in [Falc3] can be summarized as follows: Given a linear transformation $S$ on $\mathbb{R}^d$ with singular values $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_d$, the singular value function $\phi^\gamma$ is for positive $\gamma$ defined by

$$\phi^\gamma(S) = \begin{cases} \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{m-1} \cdot \alpha_m^{\gamma+1-m} & \text{if } \gamma \leq d, \\ (\alpha_1 \cdot \ldots \cdot \alpha_d)^{\gamma/d} & \text{otherwise,} \end{cases}$$

where $m = \lceil \gamma \rceil$. For a family $S_1, \ldots, S_r$ of contractive linear transformations on $\mathbb{R}^d$ let $\Delta = \Delta(S_1, \ldots, S_r)$ be the unique $\Delta > 0$ such that

$$\lim_{n\to\infty} \Big( \sum_{\underline{i}\in I_n} \phi^\Delta(S_{\underline{i}}) \Big)^{1/n} = 1.$$

**Theorem 3.5 (Falconer)** *Assume that $\| S_i \| < 1/3$ for $i = 1, \ldots, r$. Then, the unique nonempty invariant compact set*

$$K = \bigcup_{i=1}^r S_i(K) + a_i = \langle S_1(\cdot) + a_1, \ldots, S_r(\cdot) + a_r \rangle$$

*has the dimension*

$$d_{HD}(K) = d_{\text{box}}(K) = \min\left(d, \Delta(S_1, \ldots, S_r)\right)$$

*for almost all $(a_1, \ldots, a_r) \in \mathbb{R}^{rd}$ in the sense of $rd$-dimensional Lebesgue measure.*

**Remark** If $\| S_i \| < 1/3$ is not satisfied replace $\{S_1, \ldots, S_r\}$ by $\{S_{\underline{i}} : |\underline{i}| = n\}$ where $n$ is chosen so that $\| S_{\underline{i}} \| < 1/3$ for $\underline{i} \in I_n$. Since $\Delta(S_1, \ldots, S_r) = \Delta(\{S_{\underline{i}} : |\underline{i}| = n\})$ and $K$ is as well invariant under $\{S_{\underline{i}} : |\underline{i}| = n\}$ the dimension of $K$ is still given as in the theorem, but for almost all parameters $a \in \mathbb{R}^{dr^n}$ in the sense of $dr^n$-dimensional Lebesgue measure.
It is quite easy to see that $\Delta(S_1, \ldots, S_r)$ is the limes of the sequence $\Delta_n$ defined through

$$\sum_{\underline{i}\in I_n} \phi^{\Delta_n}(S_{\underline{i}}) = 1.$$

Moreover, [Falc3] actually proves that these numbers $\Delta_n$ bound $\overline{d_{\text{box}}}(K)$ from above. (This was already detected in [DO] in a slightly different context.) A comparable situation is found in our theorem 3.3 iii), where the numbers $\gamma(J_\delta)$ bound $T(q)$, and where $\underline{T}(q) = \lim \gamma(J_\delta)$.
As this relation suggests, the value $\Delta$ can be computed in a similar way as $\Gamma$. To this end set

$$\varphi_{\underline{i}}(\gamma) = \begin{cases} \lambda_{\underline{i}}^{\,\gamma} & \text{if } 0 \leq \gamma \leq 1, \\ \lambda_{\underline{i}}\nu_{\underline{i}}^{\,\gamma-1} & \text{if } 1 < \gamma \leq 2, \\ (\lambda_{\underline{i}}\nu_{\underline{i}})^{\gamma/2} & \text{if } 2 < \gamma, \end{cases} \quad \text{and} \quad \theta_{\underline{i}}(\gamma) = \begin{cases} \nu_{\underline{i}}^{\,\gamma} & \text{if } 0 \leq \gamma \leq 1, \\ \nu_{\underline{i}}\lambda_{\underline{i}}^{\,\gamma-1} & \text{if } 1 < \gamma \leq 2, \\ (\nu_{\underline{i}}\lambda_{\underline{i}})^{\gamma/2} & \text{if } 2 < \gamma. \end{cases}$$

**Theorem 3.6 (Value of $\Delta$)** *Let $S_i(x, y) = (\vartheta_i\lambda_i x, \zeta_i\nu_i y)$ $(i = 1, \ldots, r)$ be the linear parts of a set of diagonal affine contractions as in (3.1). Then*

$$\Delta(S_1, \ldots, S_r) = \max(\Delta^+, \Delta^-),$$

*where $\Delta^+$ and $\Delta^-$ are uniquely defined by*

$$\sum_{i=1}^r \varphi_i(\Delta^+) = 1 \quad \textit{resp.} \quad \sum_{i=1}^r \theta_i(\Delta^-) = 1.$$

*In particular, if $\lambda_i \geq \nu_i$ $(i = 1, \ldots, r)$, then $\Delta = \Delta^+$.*

**Remark** For the actual Hausdorff dimension of certain SAMF see (3.28) on p. 104.
**Proof** We proceed in a similar manner as in section 3.2.

**o)** First, if $\Delta^+ \geq 2$, then $\sum \lambda_i\nu_i \geq 1$, $\Delta^+ = \Delta^-$ and $\sum_{\underline{i}\in I_n} \phi^{\Delta^+}(S_{\underline{i}}) = 1$ for all $n$. Thus assume $\Delta^+ < 2$ for the remainder.

**i)** By definition, $\phi^\gamma(S_{\underline{i}})$ equals $\varphi_{\underline{i}}(\gamma)$ if $\lambda_{\underline{i}} > \nu_{\underline{i}}$, resp. $\theta_{\underline{i}}(\gamma)$ if $\lambda_{\underline{i}} \leq \nu_{\underline{i}}$. For fixed $\gamma$ define a probability space $(I_\infty, \mathcal{B}, P^+)$ where $P^+$ is the product measure on $\mathcal{B}$ induced by the measures

$$\{i\} \mapsto \frac{\varphi_i(\gamma)}{\sum_{i=1}^r \varphi_i(\gamma)}$$

on the factors $\{1, \ldots, r\}$ of $I_\infty$. Remind $c_i = \log(\lambda_i/\nu_i)$. The random variables

$$X_n : I_\infty \to \mathbb{R} \quad \underline{i}_\infty \mapsto c_{i_n}$$

are independent and identically distributed, and their common expectation amounts

$$E_\gamma[X_n] = \Big(\sum_{i=1}^{r} \varphi_i(\gamma)\Big)^{-1} \sum_{i=1}^{r} c_i\varphi_i(\gamma).$$

Defining a product measure $P^-$ and random variables $Y_n$ the same way but with $\theta$ replacing $\varphi$ and $d_i = -c_i$ replacing $c_i$, yields for fixed $\gamma$

$$\sum_{\underline{i}\in I_n} \phi^\gamma(S_{\underline{i}}) = \sum_{\underline{i}\in I_n^+} \varphi_{\underline{i}}(\gamma) + \sum_{\underline{i}\in I_n^-} \theta_{\underline{i}}(\gamma) =$$

$$\Big(\sum_{i=1}^{r} \varphi_i(\gamma)\Big)^n P^+[X_1+\ldots+X_n>0] + \Big(\sum_{i=1}^{r} \theta_i(\gamma)\Big)^n P^-[Y_1+\ldots+Y_n\geq 0]$$

(compare lemma 3.12). The way to proceed now is: assume first that $\Delta^+ \geq \Delta^-$. Then $E_{\Delta^+}[X_n] \geq 0$ by ii) below. Since the positive function $\theta_i(\cdot)$ is monotonous, one finds

$$\sqrt[n]{P^+[X_1+\ldots+X_n>0]} \leq \Big(\sum_{\underline{i}\in I_n} \phi^{\Delta^+}(S_{\underline{i}})\Big)^{1/n} \leq \sqrt[n]{2},$$

thus $\Delta^+ = \Delta(S_1,\ldots,S_r)$ by the Central Limit Theorem. Similar considerations yield $\Delta^- = \Delta(S_1,\ldots,S_r)$, provided $\Delta^- \geq \Delta^+$.

**ii)** Assume $E_{\Delta^+}[X_n] < 0$. The claim is $\Delta^+ < \Delta^-$. For the proof a 'hidden' variable $a$ has to be introduced. Therefore, modify (3.12) to

$$\chi(a,\gamma) = \sum_{i=1}^{r} {\lambda_i}^a {\nu_i}^{\gamma-a}.$$

Different cases are considered.

1. $0 < \Delta^+ \leq 1$: then $\Delta^- \leq 1$ may be assumed. By assumption
$$E_{\Delta^+}[X_n] = \frac{\chi_{.1}(\Delta^+,\Delta^+)}{\chi(\Delta^+,\Delta^+)} < 0$$
and by convexity $\chi_{.1}(a,\Delta^+) < 0$ for all $a \leq \Delta^+$. So
$$\chi(0,\Delta^+) > \chi(\Delta^+,\Delta^+) = \sum_{i=1}^{r} \lambda_i^{\Delta^+} = 1 = \sum_{i=1}^{r} \nu_i^{\Delta^-} = \chi(0,\Delta^-)$$
and $\Delta^+ < \Delta^-$.
2. $1 < \Delta^+ < 2$: this time the expectation $E_{\Delta^+}[X_n]$ has the same sign as $\chi_{.1}(1,\Delta^+)$, so $\chi_{.1}(a,\Delta^+) < 0$ for all $a \leq 1$. Assume first $\Delta^- \leq 1$. Then
$$\chi(0,\Delta^+) > \chi(1,\Delta^+) = \sum_{i=1}^{r} \lambda_i\nu_i^{\Delta^+-1} = 1 = \sum_{i=1}^{r} \nu_i^{\Delta^-} = \chi(0,\Delta^-)$$
and $\Delta^+ < \Delta^-$, which is actually a contradiction. If $\Delta^- \geq 2$ there is nothing to prove. So assume finally $1 < \Delta^- < 2$. Then
$$\chi(\Delta^- -1,\Delta^+) > \chi(1,\Delta^+) = 1 = \sum_{i=1}^{r} \nu_i\lambda_i^{\Delta^- -1} = \chi(\Delta^- -1,\Delta^-)$$
and again $\Delta^+ < \Delta^-$.



**iii)** Finally assume $\lambda_i \geq \nu_i$ $(i=1,\ldots,r)$. Then $\Delta^+ \geq \Delta^-$ since $E_\gamma \geq 0$ with equality only when $\lambda_i = \nu_i$ $(i=1,\ldots,r)$. This completes the proof. $\diamond$

The relevant values $\Delta^+$ and $\Delta^-$ are quite simpler defined than $\Gamma^+$ and $\Gamma^-$ (3.22). The reason is that, due to implicit properties, the expectations of the random variables involved have the 'good' sign. Therefore, it is not necessary to apply Chernoff's theorem. Some inherent geometrical properties allow us to simplify the formula for the box dimension of the support of a SAMF in a similar manner. This will make it possible to compare the 'sure' box dimension $D_0$ with the 'almost sure' $\Delta(S_1,\ldots,S_r)$.

**Theorem 3.7** *Let $\mu$ be a SAMF with support $K$ and assume that $D^{(k)} = d_{\text{box}}(K^{(k)})$ exists for $k = 1, 2$. Then*

$$\boxed{d_{\text{box}}(K) = \max(d^+,d^-)}$$

*where $d^+$ and $d^-$ are defined through*

$$\sum_{i=1}^{r} {\lambda_i}^{D^{(1)}} {\nu_i}^{(d^+-D^{(1)})} = 1 \quad \textit{resp.} \quad \sum_{i=1}^{r} {\nu_i}^{D^{(2)}} {\lambda_i}^{(d^--D^{(2)})} = 1$$

*In particular, if $\lambda_i \geq \nu_i$ $(i=1,\ldots,r)$, then $d_{\text{box}}(K) = d^+$.*

This formula covers results from [GL] and [Mu]. The following lemma allows an easier handling of $\Gamma$. For SMFs it is not needed since then $\gamma^+ = \gamma^- = \Gamma$ for all $q$.

**Lemma 3.18** *Assume that not all $c_i = 0$, i.e. $\mu$ is not a SMF. For convenience write $a = T^{(1)}(q)$, $b = T^{(2)}(q)$, $\gamma^+ = \gamma^+(q)$, $\gamma^- = \gamma^-(q)$ and $\Gamma = \Gamma(q)$. Then,*

$$\gamma^+ \leq a+b \Leftrightarrow \gamma^- \leq a+b \Leftrightarrow \Gamma = \max(\gamma^+,\gamma^-) \Leftrightarrow \Gamma \leq a+b.$$

**Proof** The functions $\chi$ and $\psi$ from section 3.2 will be in use.

**i)** Assume first that $\gamma^+ < a+b$. Then

$$\sum_{i=1}^{r} {\lambda_i}^a {\nu_i}^b < \sum_{i=1}^{r} {\lambda_i}^a {\nu_i}^{(\gamma^+-a)} = 1$$

and hence $\gamma^- < a+b$. Similarly $\gamma^+ > a+b$ iff $\gamma^- > a+b$. This shows the first equivalence.

**ii)** Next the easy case $\gamma^+ = \gamma^- = a+b$ is treated: by direct computation $\chi_{.1}(a,a+b) = -\psi(b,a+b)$ and so at least one of them is greater or equal to zero. This yields: $\Gamma^+ = \gamma^+$ or $\Gamma^- = \gamma^-$. Thus $\Gamma = a+b$, what was to show.

**iii)** Now take the case $\gamma^+ < a+b$ and $\gamma^- < a+b$. As will be shown at once, $\gamma^+ \geq \gamma^-$ implies $\chi_{.1}(a,\gamma^+) \geq 0$ and so $\gamma^+ = \Gamma^+ = \Gamma$. The symmetric argument for the case $\gamma^+ \leq \gamma^-$ yields indeed $\Gamma = \max(\gamma^+,\gamma^-)$ as desired. The proof is by contradiction and resembles the one in theorem 3.6: assume $\chi_{.1}(a,\gamma^+) < 0$. Then $\chi_{.1}(x,\gamma^+) < 0$ for all $x \leq a$ by convexity and

$$\chi(\gamma^- - b,\gamma^+) > \chi(a,\gamma^+) = 1 = \psi(b,\gamma^-) = \chi(\gamma^- - b,\gamma^-).$$

The monotonicity of $\chi$ in the variable $\gamma$ yields $\gamma^+ < \gamma^-$.

**iv)** Next take the case $\gamma^+ > a+b$ and $\gamma^- > a+b$. Now the aim is to show that $\Gamma$ cannot equal $\max(\gamma^+,\gamma^-)$. (This is not true for SMFs.) The idea is to prove that $\gamma^+ \geq \gamma^-$ implies $\chi_{.1}(a,\gamma^+) < 0$, and so $\Gamma^+ = \gamma_0 < \gamma^+$ by (3.19). So assume $\chi_{.1}(a,\gamma^+) \geq 0$. Strict convexity implies $\chi_{.1}(x,\gamma^+) > 0$ for all $x > a$, $\chi(\gamma^- - b,\gamma^+) > \chi(a,\gamma^+) = 1 = \chi(\gamma^- - b,\gamma^-)$ and indeed $\gamma^+ < \gamma^-$. This means in particular that if $\gamma^+ = \gamma^-$ then $\Gamma = \gamma_0$.

**v)** It remains to show, that $\gamma^+ > a+b$, $\chi_{.1}(a,\gamma^+) < 0$ and $\psi_{.1}(b,\gamma^-) < 0$ imply $\gamma_0 > a+b$. Assume the contrary, i.e. $\gamma_0 \leq a+b < \min(\gamma^+,\gamma^-)$. By lemma 3.14 ii) $\chi_{.1}(a,\cdot)$ and $\psi_{.1}(b,\cdot)$ have no zero in $[\gamma_0,\min(\gamma^+,\gamma^-)]$. Hence $0 > \chi_{.1}(a,a+b) = -\psi(b,a+b) > 0$ which is impossible. $\diamond$

**Proof of the Theorem** In the case of a SFM it is enough to refer to (2.13). For a SAMF corollary 2.1 and theorem 3.3 imply $d_{\text{box}}(K) = T(0) = \Gamma(0)$. Of course $d^+ = \gamma^+(0)$ , $d^- = \gamma^-(0)$. By lemma 3.18 it is enough to show $T(0) \leq T^{(1)}(0)+T^{(2)}(0)$, i.e. $d_{\text{box}}(K) \leq D^{(1)} + D^{(2)}$. But this is immediate since $K$ is a subset of $K^{(1)}\times K^{(2)}$. For general $q$, however, there is no a-priori inequality between $T(q)$ and $T^{(1)}(q)+T^{(2)}(q)$ (see Ex. 3.2). Finally use corollary 3.2. $\diamond$

The theorems 3.5, 3.6 and 3.7 fuse to the following considerations concerning the box dimension of self-affine sets: let $w_i(x,y) = S_i(x,y) + (u_i,v_i)$ be diagonal affine contractions (3.1) and $K$ their invariant compact set. For simplicity assume in addition that $\| S_i \| < 1/3$. The projections $K^{(k)}$ are invariant under the IFS $({w_1}^{(1)},\ldots,{w_r}^{(1)})$ by lemma 3.3 and Falconer's theorem can be applied.

- When $\sum\lambda_i \leq 1$ then $d_{HD}(K^{(1)}) = \Delta^+$ for almost all $(u_1,\ldots,u_r) \in \mathbb{R}^r$. If the latter holds then $\Delta^+ = d^+$. If, in addition, $\Delta^- \leq \Delta^+$, e.g. if $\lambda_i \geq \nu_i$ $(i = 1,\ldots,r)$, it may be concluded with no further assumption that
$$d_{HD}(K) = d_{\text{box}}(K) = \Delta(S_1,\ldots,S_r) = d_{HD}(K^{(1)})$$
since $d_{HD}(K) \geq d_{HD}(K^{(k)})$ and $d_{\text{box}}(K) \leq \Delta$.

- If $\sum\lambda_i \geq 1$ and $\sum\nu_i \geq 1$ then $d_{HD}(K^{(k)}) = 1$ for almost all $(u_1,\ldots,u_r) \in \mathbb{R}^r$ and almost all $(v_1,\ldots,v_r) \in \mathbb{R}^r$. As before $\Delta^+ = d^+$ and $\Delta^- = d^-$, and assuming the existence of a round open set one finds
$$d_{\text{box}}(K) = \Delta(S_1,\ldots,S_r).$$



Thus, the condition in proposition 4 in [Falc5] can considerably be weakened in the case of diagonal affinities. This is of course a consequence of the common invariant subspaces of the maps $w_i$.

Finally, two recent results should be mentioned. First, Falconer [Falc5] provides a lower bound $d_-$ of the Hausdorff dimension of self-affine sets. Using the notation in theorem 3.5 this bound holds for all $(a_1,\ldots,a_r) \in \mathbb{R}^{rd}$ such that the images $w_i(K)$ are mutually disjoint. As examples show, $d_-$ can be quite close to the upper bound $\Delta$ of the box dimension. On the other hand, since $d_-$ as well as $\Delta$ do not depend on the translations $a_i$, they must cover the worst cases and cannot be expected to be equal. A tedious but straightforward analysis of diagonal affinities gives an explicit formula for $d_-$ and reveals indeed, that $d_- < \Delta$ unless all $\lambda_i = \nu_i$. Secondly, Gatzouras et Lalley [GL] give explicit formulas for the exact Hausdorff and box dimensions of certain self-affine sets in the plane, which form a class of supports of SAMFs. Their result on the box dimension agrees with ours.

## 3.5 Centered Self-Affine Multifractals

Almost all of the SAMFs considered in this section are centered with self-similar projections. Consequently the formula for the particular singularity exponents provided by theorem 3.3 can be solved explicitly or can at least be treated on a computer. Many examples will be discussed such as the Sierpiński carpets and products of measures. Furthermore, theorem 3.3 is related to recent results, emphasizing its relevance in contemporary research.

### 3.5.1 Numerical Calculation

Provided the projections $\mu^{(1)}$ and $\mu^{(2)}$ of $\mu$ are self-similar, their singularity exponents are grid-regular and can be determined by solving implicit equations. Consequently, the exponents of a C-SAMF can be calculated and plotted by a computer.
For the moment assume only that $w_i^{(k)}(O) \cap w_j^{(k)}(O) \neq \emptyset$ implies $w_i^{(k)} = w_j^{(k)}$ $(k = 1, 2)$. Denote by $s$ resp. $t$ the number of distinct maps $w_i^{(1)}$ resp. $w_i^{(2)}$. Let $\{w_1^+,\ldots,w_s^+\}$ be an enumeration of $\{w_1^{(1)},\ldots,w_r^{(1)}\}$ and let $\{w_1^-,\ldots,w_t^-\}$ be the same for $\{w_1^{(2)},\ldots,w_r^{(2)}\}$. Furthermore, set

$$p_i^+ := \sum_{j\,:\,w_j^{(1)}=w_i^+} p_j, \qquad \lambda_i^+ := \mathrm{Lip}(w_i^+) \quad (i = 1,\ldots,s)$$

and

$$p_i^- := \sum_{j\,:\,w_j^{(2)}=w_i^-} p_j, \qquad \nu_i^- := \mathrm{Lip}(w_i^-) \quad (i = 1,\ldots,t).$$

With lemma 3.3

$$\mu^{(1)} = \sum_{j=1}^{r} p_j w_j^{(1)}{}_*\mu = \sum_{i=1}^{s} \Big( \sum_{j\,:\,w_j^{(1)}=w_i^+} p_j \Big) w_i^+{}_*\mu$$

what can be expressed as

$$\mu^{(1)} = \langle w_1^+, \dots, w_s^+; p_1^+, \dots, p_s^+ \rangle. \tag{3.26}$$

By assumption the interval $]0,1[$ is a basic open set for $(w_1^+, \dots, w_s^+)$ as well as for $(w_1^-, \dots, w_t^-)$ on the respective axis. Thus the projections of the measure are in fact self-similar. Corollary 2.5 with $d = 1$ yields the grid-regular $T^{(k)}(q)$:

$$\sum_{i=1}^{s} (p_i^+)^q (\lambda_i^+)^{T^{(1)}(q)} = 1 \quad \text{and} \quad \sum_{i=1}^{t} (p_i^-)^q (\nu_i^-)^{T^{(2)}(q)} = 1. \tag{3.27}$$

In particular, theorem 3.3 applies for all $q$. The three candidates $\gamma^+$, $\gamma^-$ and $\gamma_0$ can, therefore, be numerically determined as well as the relevant conditions, to obtain $\Gamma$. To this end some remarks: it may be helpful to use the variable $e^{\gamma^+}$ rather than $\gamma^+$, translating the transcendent equation into a polynomial one. Moreover, the calculation of $\gamma_0$ as well as some tests to determine $\Gamma^\pm$ can often be avoided due to lemma 3.18.

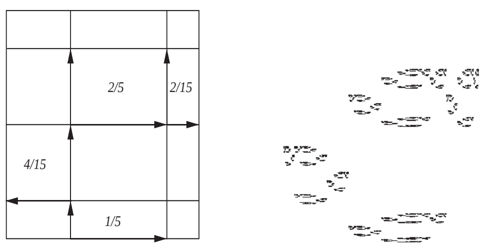


Figure 3.7: *A general self-affine multifractal with self-similar projections (Ex. 3.1). Its construction is revealed on the left. Though lemma 3.8 cannot be applied, the SAMF can be recognized as centered by considering the coarser construction by $w_{ij}([0,1]^2)$. On the right an image composed of 30'000 points obtained from a random algorithm.*

Finally, if it comes to compute $\gamma_0$, note for good first approximations of the solution $(a_0, \gamma_0)$ of (3.18) that

$$T^{(1)}(q) \le a_0 \le \gamma^- - T^{(2)}(q) \quad \text{and} \quad T^{(1)}(q) + T^{(2)}(q) \le \gamma_0 \le \min(\gamma^+, \gamma^-),$$

provided $\chi_{.1}(T^{(1)}(q), \gamma^+, q) < 0$ and $\chi_{.1}(\gamma^- - T^{(2)}(q), \gamma^-, q) > 0$. The first statement is a simple geometric fact: compare with figure 3.6, where the positions of $\gamma^+$ and $\gamma^-$ have to be interchanged due to the respective signs of $\chi_{.1}$. The second statement follows from lemma 3.18.



**Example 3.1 (A General C-SAMF)** Take

$$\begin{aligned} w_1(x,y) &= (x/2+1/3, y/6) & w_2(x,y) &= (-x/3+1/3, y/3+1/6) \\ w_3(x,y) &= (x/2+1/3, y/3+1/2) & w_4(x,y) &= (x/6+5/6, y/3+1/2). \end{aligned}$$

with the round open set $]0,1[^2$ (see Fig. 3.7). The projections $K^{(k)}$ are then self-similar. To establish a corresponding SAMF as centered, one has to consider the coarser IFS $w_{ij}$ with $ij \in \{1, \dots, 4\}^2$ and to observe that $w_2$ reverses the orientation on the $x^{(1)}$-axis.
Choose first the probabilities $p_1 = 1/5$, $p_2 = 4/15$, $p_3 = 2/5$ and $p_4 = 2/15$. Then, a numerical evaluation shows that $\Gamma = \gamma^+$ for all $q$, $D_\infty = 1.106$, $D_0 = 1.309$, $D_1 = 1.287$, $D_{-\infty} = 1.636$ and $F(D_\infty) = F(D_{-\infty}) = 0$ (see Fig. 3.8).

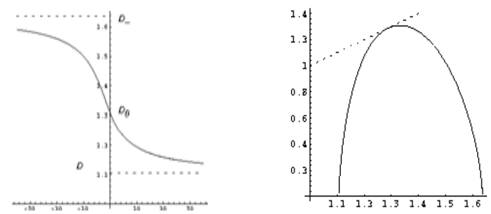

Figure 3.8: *Example 3.1 with $p_1 = 1/5$, $p_2 = 4/15$, $p_3 = 2/5$ and $p_4 = 2/15$: Generalized dimensions and spectrum, the latter obtained as a parametric plot (1.22) with $-35 \le q \le 35$. Note the slow convergence of $D_q$ to $D_\infty = 1.106$, in particular compare $D_{35} = 1.138$ with $-T'(35) = 1.108$.*

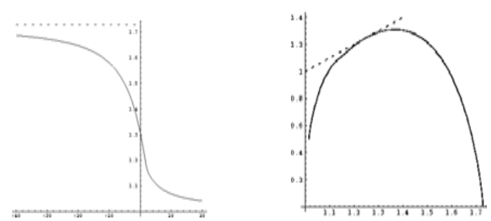

Figure 3.9: *On the left the generalized dimensions and on the right the spectrum of example 3.1 with $p_1 = p_2 = p_3 = p_4 = 1/4$. Note how the concavity of the spectrum is disturbed but not distroyed.*

Choosing then the probabilities all equal to 1/4 results in $\Gamma = \gamma^+$ for $q < 1.838$ and $\Gamma = \gamma_0$ otherwise, in particular $\Gamma = \Gamma^+$ on all of $\mathbb{R}$. Furthermore, $D_\infty = 1$, $D_1 = 1.245$, $D_{-\infty} = 1.728$, $F(D_\infty) = 1/2$ and $F(D_{-\infty}) = 0$ (see Fig. 3.9).
Thus, $T$ is in both cases continuously differentiable, and consequently the spectrum

is in fact grid-regular. Note how in the second choice of $p_i$ the change of $\Gamma$ from $\gamma^+$ to $\gamma_0$ disturbs the concavity of $F$, but does not distroy it. ○

### 3.5.2 Carpets and Subsets of Given Local Hölder Exponent

To present a first special kind of SAMFs consider a (fixed) IFS of diagonal affine contractions having the following three properties: $K^{(1)}$ is self-similar, $\lambda_i \geq \nu_i$ for $i = 1, \dots, r$, and $]0,1[^2$ is a basic open set. Denote by $\mu(\mathbf{p})$ the corresponding SAMF with probability vector $\mathbf{p}$. Then, the invariant set $K = \mathrm{supp}(\mu(\mathbf{p}))$ does not depend on $\mathbf{p}$ and the information dimension $D_1(\mathbf{p})$ of $\mu(\mathbf{p})$ is provided by corollaries 3.4 and 2.7.

**Theorem 3.8 (Gatzouras, Lalley [GL])** *With the notation from above:*

$$d_{\mathrm{HD}}(\mu(\mathbf{p})) = D_1(\mathbf{p})$$

$$d_{\mathrm{HD}}(K) = \max\{D_1(\mathbf{p}) : \ \mathbf{p} \ \text{ is a probability vector}\} \tag{3.28}$$

*Moreover,*

$$\delta = d_{\mathrm{HD}}(K) = d_{\mathrm{box}}(K) \Leftrightarrow 0 < m^\delta(K) < \infty \Leftrightarrow \sum_{j \,:\, w_j^{(1)} = w_i^+} \nu_j^{\delta - D_0{}^{(1)}} = 1 \ (i = 1, \dots, s)$$

Our contribution are the singularity exponents, in particular $d_{\mathrm{box}}(K)$ which agrees with the value found in [GL] (see theorem 3.7). Explicit formulas for $T(q)$ are obtained under almost the same conditions as above, i.e. provided the underlying IFS has the following two properties: $K^{(1)}$ is self-similar and $\lambda_i = \lambda > \nu = \nu_i$ $(i = 1, \dots, r)$. For convenience we will address the corresponding multifractals as *carpets*.

**Corollary 3.9 (Carpets)** *Let $\mu$ be a carpet. If $q < 0$ assume in addition that $\mu$ is vertically centered. Then $T(q)$ is grid-regular and, with the notation of (3.26),*

$$T(q) = \Big(\frac{1}{\log \nu} - \frac{1}{\log \lambda}\Big) \log\Big(\sum_{i=1}^{s} (p_i^+)^q\Big) - \frac{1}{\log \nu} \log\Big(\sum_{i=1}^{r} p_i{}^q\Big).$$

**Proof** Apply corollary 3.2 and use that $\mu^{(1)}$ is self-similar (3.26). ◇

There is a number of notable values of $D_q$. With (3.26), the box dimension of a carpet is

$$D_0 = d_{\mathrm{box}}(K) = \frac{\log s}{\log(1/\lambda)} + \frac{\log(r/s)}{\log(1/\nu)}. \tag{3.29}$$

This formula was already found by McMullen [Mu] and Bedford [Bed1] for a special kind of carpets, the so-called *generalized Sierpiński carpets*.



Furthermore, we take the chance having explicit formulas for a study of the 'most probable' resp. 'most rarefied' points. By straightforward calculation ($\nu_i = \nu$):

$$\begin{aligned} D_\infty &= \Big(\frac{1}{\log \lambda} - \frac{1}{\log \nu}\Big) \log\Big(\max(p_1^+, \dots, p_s^+)\Big) + \frac{1}{\log \nu} \log\Big(\max(p_1, \dots, p_r)\Big) \\ D_{-\infty} &= \Big(\frac{1}{\log \lambda} - \frac{1}{\log \nu}\Big) \log\Big(\min(p_1^+, \dots, p_s^+)\Big) + \frac{1}{\log \nu} \log\Big(\min(p_1, \dots, p_r)\Big), \end{aligned}$$

and

$$\begin{aligned} F(D_\infty) &= \Big(\frac{1}{\log \nu} - \frac{1}{\log \lambda}\Big) \log(u^+) - \frac{1}{\log \nu} \log(u) \\ F(D_{-\infty}) &= \Big(\frac{1}{\log \nu} - \frac{1}{\log \lambda}\Big) \log(v^+) - \frac{1}{\log \nu} \log(v), \end{aligned}$$

where $u^{(+)}$ resp. $v^{(+)}$ denote the number of maximal resp. minimal $p_i^{(+)}$.
In order to give a heuristic argument consider a cylindrical set $V_{\underline{i}}$ ($\underline{i} \in J_{\delta_n}$). This is essentially a $\lambda_{\underline{i}} \times \nu_{\underline{i}}$-rectangle and can be subdivided into squares of side $\delta_n = \nu_{\underline{i}}$. The $k$-th square, counting according to the orientation induced by $w_{\underline{i}}$, has the measure $p_{\underline{i}} \cdot \mu^{(1)}([(k-1)\nu_{\underline{i}}/\lambda_{\underline{i}}, k\nu_{\underline{i}}/\lambda_{\underline{i}}[)$. But $\mu^{(1)}$ is a SMF, thus $D_\infty{}^{(1)} = \min(\log p_i^+ / \log \lambda_i^+)$ and the $\mu^{(1)}$-measure of an $\varepsilon$-box on $\mathbb{R}$ amounts at the most $\varepsilon^{D_\infty{}^{(1)}}$, i.e. when the box coincides with some $w_{\underline{j}}{}^{(1)}([0,1])$, where $\log p_{j_k}^+ / \log \lambda_{j_k}^+ = D_\infty{}^{(1)}$ for all letters $j_k$ of $\underline{j}$.
Choosing $\varepsilon = \nu_{\underline{i}}/\lambda_{\underline{i}}$ it can be seen that the measure of a $\delta_n$-square $B$ contained in $V_{\underline{i}}$ amounts at the most

$$\mu(B) = p_{\underline{i}} \cdot (\nu_{\underline{i}}/\lambda_{\underline{i}})^{D_\infty{}^{(1)}}.$$

The number of such squares is approximately $(\lambda_{\underline{i}}/\nu_{\underline{i}})^{F_\infty^{(1)}}$ where $F_\infty^{(1)} := F^{(1)}(D_\infty{}^{(1)})$. Thus, taking the squares for $\delta_n$-boxes, $\mu(B) = \delta_n{}^{D_\infty}$ is only possible if

$$p_i = \lambda_i{}^{D_\infty{}^{(1)}} \nu_i{}^{D_\infty - D_\infty{}^{(1)}} \tag{3.30}$$

holds, where $i$ passes through all letters $i_k$ of $\underline{i}$. (Otherwise, i.e. if (3.30) would not hold for all $i_k$, there had to exist a letter $l$ with $p_l$ greater than the right-hand side of (3.30). Considering the squares in $V_{\underline{i}}$ for $\underline{i} = l * \dots l$ reveals that $D_\infty$ would not be the smallest Hölder exponent in contradiction to theorem 1.2. Consequently

$$\lim_{n\to\infty} \frac{\log \#\{B \in G_{\delta_n} \,:\, \mu(B) = \delta_n{}^{D_\infty}\}}{-\log \delta_n} = \gamma \tag{3.31}$$

where

$$\sum \lambda_i{}^{F_\infty^{(1)}} \nu_i{}^{\gamma - F_\infty^{(1)}} = 1, \tag{3.32}$$

the sum being taken over all $i$ satisfying (3.30). Though the argumentation is not rigorous in general, *it is certainly exact for carpets with* $\log \nu_i / \log \lambda_i = \text{const} \in \mathbb{Q}$, choosing $\delta_n$ to be a convenient power of $\nu$. Similar for $-\infty$.
Often $\gamma = F(D_\infty)$ is the solution of (3.32). Although (3.31) gives then a satisfactory interpretation of $F(D_\infty)$, it is important to note the following:

- The assumption that the measure is ordered was essentially used to derive a simple formula for $\gamma$. For an ordered C-SAMFs see examples 3.3 and 3.6.
- The equation (3.31) describes merely a fact concerning numbers of boxes with particular properties. It says nothing about the dimension of a set whatsoever. Compare with example 3.3.

Now let us check the formulas for carpets. Straightforward calculation yields ${D_\infty}^{(1)}$ $= \log(\max(p_i^+))/\log \lambda$ and $F_\infty^{(1)} = -\log u^+/\log \lambda$, and thus indeed

$$\lambda_i^{{D_\infty}^{(1)}} \nu_i^{D_\infty - {D_\infty}^{(1)}} = \max(p_i)$$

and

$$\lambda_i^{F_\infty^{(1)}} \nu_i^{F(D_\infty) - F_\infty^{(1)}} = \frac{1}{u^+} \cdot \frac{u^+}{u} = 1/u$$

for all $i$. So (3.30) chooses all maximal probabilities $p_i$ and (3.32) is indeed satisfied by $\gamma = F(D_\infty)$. Provided the $u$ maps $w_{i_l}$ with the maximal probabilities are arranged in the $u^+$ columns corresponding to the maximal $p_i^+$, then their invariant set $K' = \langle w_{i_1}, \ldots, w_{i_u} \rangle$ may fairly be called the set of most probable points. Moreover, its box dimension is just $F(D_\infty)$ due to (3.29). The similar holds replacing $\infty$ by $-\infty$. However, if the maps are arranged in a different manner it is not as simple to give the 'dense' parts of $\mu$. So the interpretation of $F(\alpha)$ as the dimension of subsets $K_\alpha$ with 'local Hölder exponent $\alpha$' (compare Ex. 2.10) has to be carried out cautiously. On the other hand, it has to be referred to [S]. There it is proven that under restricted circumstances $F(\alpha)$ equals indeed the Hausdorff dimension of $K_\alpha$ 'almost surely' in the sense of theorem 3.5.

**Example 3.2 (Three Carpets)** Take

$$w_i(x, y) = (x/3, y/4) + (u_i, v_i) \quad (i = 1, \ldots, 6)$$

with the following entries $(u_i, v_i)$ in rising order

$$(0, 0) \quad (2/3, 1/4) \quad (0, 2/4) \quad (2/3, 2/4) \quad (0, 3/4) \quad (2/3, 3/4).$$

The IFS is centered with respect to the open set $O' = ]-1/2, 3/2[\times]0, 1[$ (see Fig. 3.10). Choosing $p_1 = p_2 = 1/4$, $p_3 = \ldots = p_6 = 1/8$ gives $p_i^+ = 1/2$, $p_i^- = 1/4$,

$$D_q^{(1)} \equiv \frac{\log 2}{\log 3} \qquad D_q^{(2)} \equiv 1$$



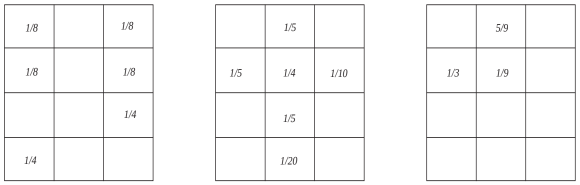


Figure 3.10: *The construction of the three carpets of example 3.2. The IFS on the left is centered with respect to the round open set* $]-1/2, 3/2[\times]0, 1[$, *the others as usual with respect to the unit square.*

and

$$T(q) = \Big(\frac{\log 2}{\log 3} - \frac{1}{2}\Big)(1 - q) + \frac{1}{\log 4} \log\Big(2^{2-3q} + 2^{1-2q}\Big).$$

Hence:

- Although $\mu^{(1)}$ and $\mu^{(2)}$ are both homogeneous $\mu$ is not (see Fig. 3.11).

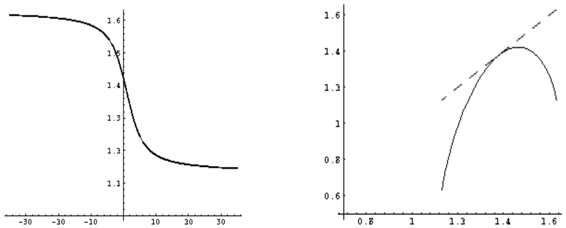

Figure 3.11: *On the left the generalized dimensions and on the right the spectrum of the first carpet appearing in example 3.2.*

A different choice of probabilities $p_i$ shows that there is a priori no inequality between $T(q)$ and $T^{(1)}(q) + T^{(2)}(q)$ unless $q = 0$ (compare theorem 3.7): corollary 3.9 may be rewritten as

$$T(q) = T^{(1)}(q) + T^{(2)}(q) + \frac{\log\Big(\sum_{i=1}^s (p_i^+)^q \sum_{i=1}^t (p_i^-)^q\Big) - \log\Big(\sum_{i=1}^r {p_i}^q\Big)}{\log \nu}.$$

Taking $w_i$ as above but with the following entries $(u_i, v_i)$ in rising order

$$(1/3, 0) \quad (1/3, 1/4) \quad (0, 2/4) \quad (1/3, 2/4) \quad (2/3, 2/4) \quad (1/3, 3/4)$$

and with $p_1 = 1/20$, $p_2 = 1/5$, $p_3 = 1/5$, $p_4 = 1/4$, $p_5 = 1/10$, $p_6 = 1/5$ one obtains $T(q) < T^{(1)}(q) + T^{(2)}(q)$ for large $|q|$. Finally taking $p_1 = p_2 = p_5 = 0$ (i.e. omitting the corresponding maps) and $p_3 = 1/3$, $p_4 = 1/9$ and $p_6 = 5/9$ leads to $T(q) > T^{(1)}(q) + T^{(2)}(q)$ for large $|q|$. Thus

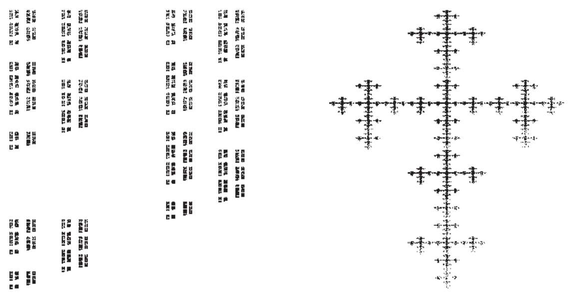

Figure 3.12: *Images of the first two carpets appearing in example 3.2, provided by a random algorithm.*

- The only a priori inequality between $T(q)$ and $T^{(1)}(q)+T^{(2)}(q)$ is at $q=0$.

$\bigcirc$

### 3.5.3 Products

The C-SAMFs of the second special type are not necessarily ordered but still possess a simple expression for their generalized dimensions and their spectrum: the product of two self-similar multifractals on $\mathbb{R}$. Such measures are considered in [Z].

**Corollary 3.10 (Product Measures)**
*Let* $\mu^{+(-)} := \langle w_1^{+(-)},\ldots,w_{s(t)}^{+(-)}; p_1^{+(-)},\ldots,p_{s(t)}^{+(-)}\rangle$ *be two given SMFs on* $\mathbb{R}$. *Define*

$$w_{(ij)}(x,y) := (w_i^+(x), w_j^-(y)) \quad p_{(ij)} := p_i^+ p_j^- \tag{3.33}$$

*for all pairs* $(ij)\in\{1,\ldots,s\}\times\{1,\ldots,t\}\sim\{1,\ldots,r\}$. *Then*

$$\mu := \langle w_{(11)},\ldots,w_{(st)}; p_{(11)},\ldots,p_{(st)}\rangle$$

*is the product measure of* $\mu^+$ *and* $\mu^-$. *Moreover,* $\mu$ *is a C-SAMF,* $\mu^{(1)}=\mu^+$, $\mu^{(2)}=\mu^-$,

$$T(q)=T^{(1)}(q)+T^{(2)}(q), \qquad D_q = {D_q}^{(1)}+{D_q}^{(2)}$$

*and*

$$F(\alpha^{(1)}+\alpha^{(2)}) = F^{(1)}(\alpha^{(1)})+F^{(2)}(\alpha^{(2)})$$

*for* $\alpha^{(k)} = -(T^{(k)})'(q)$. *Finally note that* $T$ *and* $F$ *are grid-regular.*

**Remark** The relation $T(q)=T^{(1)}(q)+T^{(2)}(q)$ holds of course for general products of multifractals, and if $\mu^+$ and $\mu^-$ are CMFs then $\mu$ is one too. For more information about the dimension of the product of metric spaces we propose [W] and [Tr].



**Proof** Instead of giving the straightforward general proof of the additivity of generalized dimensions theorem 1.1 shall be applied, for it provides also the grid-regularity. Certainly ${w_{(ij)}}^{(1)} = w_i^+$, ${w_{(ij)}}^{(2)} = w_j^-$ and

$$\sum_{(ij)\,:\,w_{(ij)}=w_k^+} p_{(ij)} = \sum_{j=1}^{t} p_k^+ p_j^- = p_k^+.$$

Thus $\mu^{(1)}=\mu^+$ by (3.26) and lemma 3.3. Similarly $\mu^{(2)}=\mu^-$. Almost evidently $\mu$ is the product measure of $\mu^{(1)}$ and $\mu^{(2)}$ by construction. Straightforward computation yields $\gamma^+=\gamma^-=T^{(1)}+T^{(2)}$. Lemma 3.18 completes the proof. $\diamond$

### 3.5.4 A Further Explicit Formula

A third kind of C-SAMFs with explicit formula for $T(q)$ is the following.
Assume that the projections $K^{(k)}$ are self-similar, furthermore, that $\nu=\lambda^2$ and that

$$\begin{array}{rclrcll} \lambda_i &=& \lambda, & \nu_i &=& \nu & (i=1,\ldots,r_1)\\ \lambda_i &=& \nu, & \nu_i &=& \lambda & (i=r_1+1,\ldots,r) \end{array}$$

for some integer $r_1$ strictly between 0 and $r$. Without loss of generality it is possible to write

$$\begin{array}{rcll rcll} \lambda_i^+ &=& \lambda & (i=1,\ldots,s_1) & \lambda_i^+ &=& \nu & (i=s_1+1,\ldots,s)\\ \nu_i^- &=& \lambda & (i=1,\ldots,t_1) & \nu_i^- &=& \nu & (i=t_1+1,\ldots,t) \end{array}$$

for some integers $s_1<s$ and $t_1<t$. Solving equations of second order gives

$$\lambda^{T^{(1)}} = \frac{\sqrt{\left(\sum_{i=1}^{s_1}(p_i^+)^q\right)^2 + 4\sum_{i=s_1+1}^{s}(p_i^+)^q} - \sum_{i=1}^{s_1}(p_i^+)^q}{2\sum_{i=s_1+1}^{s}(p_i^+)^q}$$

and similar for $T^{(2)}(q)$. Furthermore,

$$\lambda^{\gamma^+} = \frac{\sqrt{\left(\sum_{i=r_1+1}^{r}{p_i}^q\lambda^{T^{(1)}}\right)^2 + 4\sum_{i=1}^{r_1}{p_i}^q\lambda^{-T^{(1)}}} - \sum_{i=r_1+1}^{r}{p_i}^q\lambda^{T^{(1)}}}{2\sum_{i=1}^{r_1}{p_i}^q\lambda^{-T^{(1)}}}$$

Moreover,

$$\gamma_0 = \frac{-1}{\log(\lambda\nu)}\log\left(4\Big(\sum_{i=1}^{r_1}{p_i}^q\Big)\Big(\sum_{i=r_1+1}^{r}{p_i}^q\Big)\right).$$

Finally, the tests whether $\Gamma^+=\gamma^+$ resp. $\Gamma^-=\gamma^-$ reduce to

$$\sum_{i=1}^{r_1}{p_i}^q \geq 2\lambda^{3T^{(1)}}\Big(\sum_{i=r_1+1}^{r}{p_i}^q\Big)^2 \quad \text{resp.} \quad \sum_{i=r_1+1}^{r}{p_i}^q \geq 2\lambda^{3T^{(2)}}\Big(\sum_{i=1}^{r_1}{p_i}^q\Big)^2. \tag{3.34}$$

**Example 3.3 (A 'Circular' Multifractal)** Consider the IFS

$$\begin{array}{lllllll} w_1(x,y) & = & (x/2+1/4,y/4) & & w_2(x,y) & = & (x/4+3/4,y/2+1/4) \\ w_3(x,y) & = & (x/2+1/4,y/4+3/4) & & w_4(x,y) & = & (x/4,y/2+1/4) \end{array}$$

supplied with $p_1 = \ldots = p_4 = 1/4$. It has self-similar projections $K^{(k)}$ with $p_i^+ = \lambda_i^+$, $p_i^- = \nu_i^-$ ($i = 1, 2, 3$). Thus $\mu^{(1)}$ and $\mu^{(2)}$ are homogeneous SMFs, and lemma 3.8 applies to show that $\mu$ is centered (see Fig. 3.13). Observing $d_{\text{box}}(K^{(k)}) = 1$, one

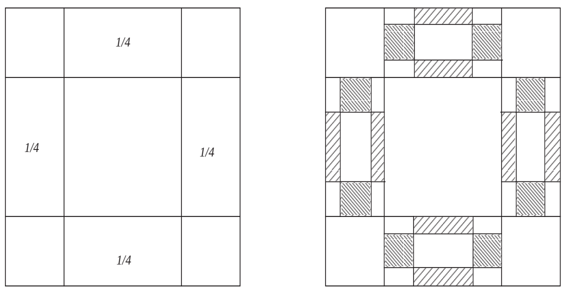


Figure 3.13: *The first two iterates in the construction of the 'circular' multifractal (see Ex. 3.3).*

obtains the (grid-regular) singularity exponents $T^{(1)}(q) = T^{(2)}(q) = 1 - q$. From this $\gamma^+ = \gamma^-$. The particular tests in the definition $\Gamma^+$ and $\Gamma^-$ coincide and read as $\gamma^+ \leq 2T^{(1)} = T^{(1)} + T^{(2)}$ by direct computation, or equivalently as $1 \geq 2^{q-1}$ using the reduced form (3.34). All this allows the conclusion

$$T(q) = \begin{cases} \gamma^+ = 3 - 2q - \log\left(\sqrt{1 + 2^{4-q}} - 1\right)/\log 2 & \text{if } q \leq 1 \\ \gamma_0 = 4/3 \cdot (1-q) & \text{otherwise.} \end{cases}$$

In particular

- The grid-regular $T$ is $C^1$ but not $C^2$ (see Fig. 3.14).
- $\Gamma = \gamma_0 \Leftrightarrow \gamma^+ > T^{(1)} + T^{(2)} \Leftrightarrow q > 1$ (compare lemma 3.18).
- The (grid-regular) spectrum $F$ comes to a sudden stop at $\alpha = D_1 = D_\infty = 4/3$. This rises the question, where the 'most probable points' can be found.

We take the opportunity and calculate explicitly. Let $\delta_n = \nu^n$ and take $\underline{i} \in J_{\delta_n}^+$, i.e. $\lambda_{\underline{i}} \geq \nu_{\underline{i}} = \nu^n$. Due to the special entries of the maps $w_i$ the set $V_{\underline{i}}$ fits into the $\delta_n$-grid. Subdividing it into squares of side $\nu^n$ yields indeed $\delta_n$-boxes. On the other hand, every $\delta_n$-box with nonvanishing measure lies in some $V_{\underline{j}}$ with $\underline{j} \in J_{\delta_n}$. Moreover, since $\mu^{(1)}$ and $\mu^{(2)}$ are homogeneous and measure just the length in $[0,1]$, all boxes



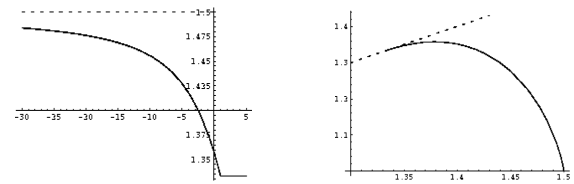

Figure 3.14: *On the left the generalized dimensions and on the right the spectrum of the circular multifractal (Ex. 3.3).*

in $V_{\underline{i}}$ have the same measure. Letting $m = |\underline{i}|$ and $k = \#\{l \leq m \,:\, i_l = 1 \text{ or } 3\}$ one finds $m + k = 2n$ due to $\nu_{\underline{i}} = \nu^n$ and

$$\mu(B) = p_{\underline{i}}(\nu_{\underline{i}}/\lambda_{\underline{i}}) = 2^{-2n-k}$$

for every $\delta_n$-box in $V_{\underline{i}}$. Observing $\lambda_{\underline{i}} \geq \nu_{\underline{i}}$ this value is found to be maximal for $k = m/2 = 2n/3$. But then $\lambda_{\underline{i}} = \nu_{\underline{i}}$ and $V_{\underline{i}}$ is a $\delta_n$-box by itself. Since $\underline{i}$ was arbitrary

$$\max\{\mu(B) \,:\, B \in G_{\delta_n}\} = 2^{-8n/3} = (\delta_n)^{4/3} = {\delta_n}^{D_\infty}.$$

There are as many $\delta_n$-boxes of this kind as words $\underline{i}$ with $\lambda_{\underline{i}} = \nu_{\underline{i}} = \nu^n$. Observing $m = 4n/3$, Stirling's formula leads us to

$$\frac{\log \#\{B \in G_{\delta_n} \,:\, \mu(B) = {\delta_n}^{D_\infty}\}}{-\log \delta_n} = \frac{\log\left(2^m \binom{m}{m/2}\right)}{n \log 4} \simeq \frac{\log\left(2^m \sqrt{2\pi/m}\, 2^m\right)}{n \log 4},$$

which converges to $4/3 = F(D_\infty)$. This agrees with the intuitive understanding of $F(D_\infty)$. But the equations (3.30) and (3.32) are not satisfied. To get a better understanding where the 'most probable' points lie, follow the words $\underline{i}$ which contribute to the 'heavy boxes' above: at the first stage of the construction the C-SAMF seems to be perfectly symmetric giving no reason for more or less probable parts. Only when regarding $\mu$ as the invariant measure $\langle w_{\underline{j}}; p_{\underline{j}} \rangle_{|\underline{j}|=t}$, differences appear and the equation (3.30) selects exactly the words $\underline{j}$ with $\lambda_{\underline{j}} = \nu_{\underline{j}}$. See figure 3.13 for $t = 2$. The equation (3.32) reads then as

$$\sum \lambda_{\underline{j}}^{F_\infty^{(1)}} \nu_{\underline{j}}^{\gamma_t - F_\infty^{(1)}} = 2^t \binom{t}{t/2} \left(8^{t/2}\right)^{\gamma_t} = 1$$

and $\gamma_t \to F(D_\infty)$ as above. Moreover, one may consider the invariant sets $L_t$ ($t$ even) generated by the IFS $\{w_{\underline{j}} \,:\, |\underline{j}| = t, \lambda_{\underline{j}} = \nu_{\underline{j}}\}$. These sets are in fact self-similar and of dimension $\gamma_t$ by (2.13). The union $K'$ of the increasing sequence of compact sets $L_t$ has the dimension

$$d_{HD}(K') = \sup_{t \in \mathbb{N}} d_{HD}(L_t) = F(D_\infty).$$

From the considerations above it is fair to call $K'$ the set of the 'most probable' points. Note in addition that the points of $K'$ certainly possess the local Hölder exponent $D_\infty$.
Turning now to $D_{-\infty}$ the minimal measure of a $\delta_n$-box is found for $k = m = n$, i.e. when $\underline{i} \in \{1,3\}^n$:

$$\min\{\mu(B) \,:\, B \in G_{\delta_n}\} = 2^{-3n} = (\delta_n)^{3/2} = \delta_n{}^{D_{-\infty}}.$$

Applying the symmetric arguments for $\lambda_{\underline{i}} < \nu_{\underline{i}}$, there are $2 \cdot 2^n (\lambda/\nu)^n = 2\delta_n{}^{-1}$ boxes of this kind, yielding

$$\frac{\log \#\{B \in G_{\delta_n} \,:\, \mu(B) = \delta_n{}^{D_{-\infty}}\}}{-\log \delta_n} \to 1 = F(D_{-\infty}).$$

On the other hand, these boxes form a decreasing sequence of compact sets converging to $K'' = \langle w_1, w_3 \rangle \cup \langle w_2, w_4 \rangle$. Certainly $K''$ is the set of all points $x$ for which the unique $\delta_n$-boxes $B_n(x)$ containing $x$ have *exactly* the measure $\delta_n{}^{D_{-\infty}}$ for all $n$. This is a strict requirement and does not exactly match the definition of local Hölder exponent $D_{-\infty}$: in the limit too strict and in the geometry too loose. As the union of two self-affine (and even self-similar) fractals, $K''$ has the dimension

$$d_{HD}(K'') = d_{\text{box}}(K'') = 1/2.$$

What seems to be a contradiction to the value $F(D_{-\infty}) = 1$ is readily explained. Since $K''$ is captivated in two line segments most of the boxes appearing in its construction above do not intersect it at all. Still they have the desired property to contribute to $F(D_{-\infty})$.

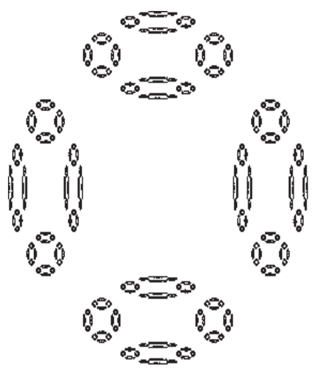

Figure 3.15: *The circular multifractal (Ex. 3.3) provided by a random algorithm.*

○

This example gives credit to the intuition that the measure is better concentrated in squares than in thin rectangles with the same measure and the same area.



### 3.5.5 Homogeneous Self-Affine Multifractals

A first and simple group of homogeneous SAMFs are the products of two homogeneous multifractals on $\mathbb{R}$. This is immediate from corollary 3.10. In particular in this case

$$p_k = \lambda_k^{D_0{}^{(1)}} \nu_k^{D_0{}^{(2)}}$$

for all $k$, analogous to the self-similar case. However, this condition is not necessary for a SAMF to be homogeneous, as the next example will show.

**Example 3.4 (A Homogeneous SAMF)** Take

$$w_i(x, y) = (x/3, y/4) + (u_i, v_i) \quad (i = 1, \ldots, 6)$$

with the following entries $(u_i, v_i)$ in rising order

$$(0, 0) \quad (2/3, 0) \quad (0, 2/4) \quad (1/3, 2/4) \quad (2/3, 2/4) \quad (1/3, 3/4).$$

This IFS has self-similar projections and lemma 3.8 applies. Choosing $p_i = 1/6$ results in $p_i^+ = 1/3$, $p_1^- = 1/3$, $p_2^- = 1/2$, $p_3^- = 1/6$,

$$T^{(1)}(q) = 1 - q \qquad T^{(2)}(q) = \frac{\log\left(3^{-q} + 2^{-q} + 6^{-q}\right)}{\log 4}$$

and the grid-regular

$$T(q) = \frac{3}{2}(1 - q).$$

○

From this example and an earlier one (3.2):

- A homogeneous SAMF need not be a product measure.
- The projections of a homogeneous SAMF do not have to be homogeneous. On the other hand, both projections may be homogeneous but $\mu$ itself not.

### 3.5.6 Applications

Now that our theory has been developed we would like to present contemporary research which is closely related. We content ourself with a narrative language and close with an interesting example of our own.
First, [Ma2] presents results of the same kind as our theorem 3.7: The box dimension of the graph of certain vector valued functions is obtained as the maximum of numbers 'measuring' the graph in different directions. The technics used for the proof are different from ours, using essentially the connectedness of the invariant set.

Further, [Koh, §4] rises the question of differentiability and tries to explain irregularities. Our examples and proposition 3.17 give some answers.
The work of Falconer [Falc3, Falc5] was already mentioned. Here we contribute a simple formula for the 'almost sure' dimension (see theorem 3.6).
Furthermore, corollary 3.9 is related to [Mu, Bed1, GL].
Remember also section 2.4 which embeds the results of chapter 2 in the field of recent publications.
Finally our theorem 3.3 covers parts of recent results, i.e. it provides the box dimension of investigated invariant sets, which are in many cases special kinds of C-SAMFs.
One such family arises from fractal interpolation: in [Bar] a new kind of interpolation functions was introduced: given $N+1$ data points $(x_i, y_i)$ one may consider a set of affine maps

$$w_i(x,y) := \begin{pmatrix} a_i & 0 \\ b_i & c_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d_i \\ e_i \end{pmatrix} \qquad i = 1 \dots N$$

with $w_i(x_0, y_0) = (x_{i-1}, y_{i-1})$ and $w_i(x_N, y_N) = (x_i, y_i)$. The compact invariant set $G$ of this IFS is then the graph of a continuous function which interpolates the data. Referring to section 2.2 an illuminating picture is provided by considering the straight line segment joining $(x_0, y_0)$ with $(x_N, y_N)$ and its iterated images under the set map $W$. Only four entries of $w_i$ are determined by the condition above. Usually $c_i$ is considered as a parameter controlling the dimension of $G$ [BEHM]:
*If* $\sum_{i=1}^{N} |c_i| > 1$ *and the interpolation points do not lie on a straight line, then* $d_{\text{box}}(G)$ *is the unique solution* $D$ *of*

$$\sum_{i=1}^{N} |c_i| {a_i}^{D-1} = 1,$$

*otherwise* $d_{\text{box}}(G) = 1$.
However, if $y_N \neq y_0$ and $|y_i - y_{i-1}| < |y_N - y_0|$ one may choose $b_i = 0$, freezing the $c_i$. Then the affinities are diagonal and the IFS is vertically centered. Moreover, ${D_0}^{(1)} = {D_0}^{(2)} = 1$ and hence $d^+ = D$ and $d^- = 1$. In this situation

$$d_{\text{box}}(G) = \Delta(w_1, \dots, w_N) = \max(1, D)$$

without any further assumptions on the geometrical situation, in agreement with theorem 3.7.
While we are able to give the generalized dimensions of $G$, many authors relate $d_{\text{box}}(G)$ to other constants describing the geometry of $G$, such as the Hölder exponent [Bed3], the Hausdorff dimension [BU, K, U, GM] or the topological pressure [Bed4, Bed2]. Some among them confine the diversity of $w_i$ to C-SAMFs by setting $b_i = 0$ [GM, GH1, Ma1, Ma2, Ma3, BU].



**Example 3.5 (Bold Play)** This example is a particular self-affine interpolation function appearing in probability theory .
Imagine a game where the chance to double the stake is $p$ and the probability to loose it is $1-p$. Then $M(x)$ represents the probability to win the amount 1 starting with the capital $x$ and observing the following strategy: if the momentary stake is $x < 1/2$, then the player wagers all, if $x \geq 1/2$ he wagers $1-x$. What makes $M(x)$ interesting is its comparison with the probability to win with 'timid play', which means to wager the same small amount $\varepsilon$ each time. While the latter promises a rather long stay at the casino, the chance to leave the place with the desired amount of money is incredibly larger when applying the first one [Fed, p 190]: betting 'black' in ordinary American roulette, $p$ is given as $18/38$. Trying to reach the goal of \$20'000 with an initial capital of \$100 by wagering \$1 each time, the probability of success is approximately $3 \cdot 10^{-911}$, wagering less 'timidly' \$10 each time increases the chance to $10^{-91}$—still completely negligible. But the chance to win with 'bold play' is 0.003!
However, the function $M$ satisfies

$$M(x) = \begin{cases} p \cdot M(2x) & \text{if } x < 1/2, \\ p + (1-p) \cdot M(2x-1) & \text{if } x \geq 1/2, \end{cases}$$

and its graph $G$ is invariant under the two diagonal contractions $w_1(x,y) = (x/2, p \cdot y)$, $w_2(x,y) = (x/2 + 1/2, (1-p)y + p)$. The IFS is centered vertically as well as horizontally with respect to the open sets $]0,1[\times]-1,3[$ and $]-1,3[\times]0,1[$. Supplying the IFS with the 'natural' probabilities $p_1 = p$, $p_2 = 1-p$ leads to

$$\begin{aligned} \gamma^+ &= T^{(1)}(q) = \log\left(p^q + (1-p)^q\right) / \log 2 \\ \gamma^- &= T^{(2)}(q) = 1 - q \end{aligned}$$

Thus $\gamma^+ \leq T^{(1)}(q) + T^{(2)}(q)$ iff $T^{(2)}(q) \geq 0$, hence iff $q \leq 1$. Furthermore, $\gamma^+ \leq \gamma^-$

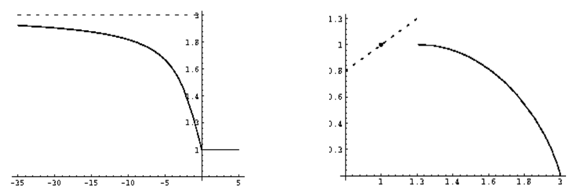

Figure 3.16: *On the left the generalized dimensions and on the right the spectrum of the graph of the 'bold play'-function with $p = 1/4$ (see Ex. 3.5).*

iff $q \in [0,1]$. The test concerning $\Gamma^-$ reduces to $p \log p + (1-p)\log(1-p) + \log 2 \geq 0$, which is true independently of $q$. So lemma 3.18 gives

$$T(q) = \left\{ \begin{array}{ll} \max(\gamma^+, \gamma^-) & \text{if } q \leq 1, \\ \min(\gamma^+, \gamma^-) & \text{if } q > 1, \end{array} \right\} = \left\{ \begin{array}{ll} \gamma^+ & \text{if } q \leq 0, \\ \gamma^- & \text{if } q > 0. \end{array} \right.$$

Since $T$ is not differentiable at zero there is a gap in the $\alpha$-interval where the spectrum is not determined by $T$.
The invariant set $G$ is the graph of the continuous function $M$ and of total length 2. Moreover, $M$ is almost everywhere differentiable with slope zero [Fed, p. 191]. For $p = 1/2$ the spectrum reduces to a point and $M$ to a linear function. In this sense $F(\alpha)$ reveals the 'nondifferentiability' of the invariant function $M$. This is due to the choice $p_i = \nu_i$, interweaving geometry and measure. $\bigcirc$

The last example provides an ordered SAMF for which $T(q)$ can explicitly be calculated for $q \geq 0$, although its projections are not self-similar.

**Example 3.6 (Rosette)**
Consider the maps

$$\begin{aligned} w_1(x,y) &= (x/2-1/2, y/4) & w_2(x,y) &= (x/2, y/2-1/2) \\ w_3(x,y) &= (x/2+1/2, y/4) & w_4(x,y) &= (x/2, y/2+1/2) \end{aligned}$$

with the round set $O = \{(x,y) : |x|+|y| < 1\}$. Choose $p_1 = p_3 = 1/4$, $p_2 = p$ and $p_4 = 1/2 - p$.

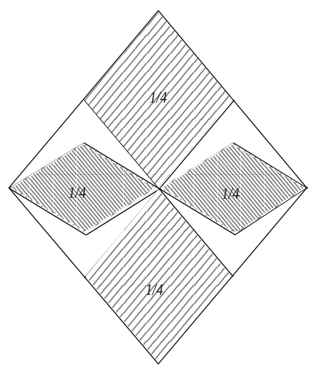


Figure 3.17: *The construction of the rosette (see Ex. 3.6).*

First $\mu^{(1)}$ is investigated. Since ${w_2}^{(1)}$ and ${w_4}^{(1)}$ coincide lemma 3.3 implies

$$\mu^{(1)} = \langle {w_1}^{(1)}, {w_2}^{(1)}, {w_3}^{(1)}; p_1, p_2+p_4, p_3 \rangle = \langle \frac{x-1}{2}, \frac{x}{2}, \frac{x+1}{2}; 1/4, 1/2, 1/4 \rangle.$$

This allows the calculation of $T^{(1)}$ although the construction of $\mu^{(1)}$ is overlapping. Denoting the normalized Lebesgue measure of $\mathbb{R}^2$ restricted to $\overline{O}$ by $\overline{\mu}$, $\mu^{(1)}$ is recognized as the projection ${\pi^{(1)}}_*\overline{\mu}$ of $\overline{\mu}$: just note that $\overline{\mu}$ is invariant under $\overline{w_i}(x,y) = (x/2, y/2) + (u_i, v_i)$ with entries $(u_i, v_i) = (-1/2, 0)$, $(0, -1/2)$, $(1/2, 0)$, $(0, 1/2)$ and



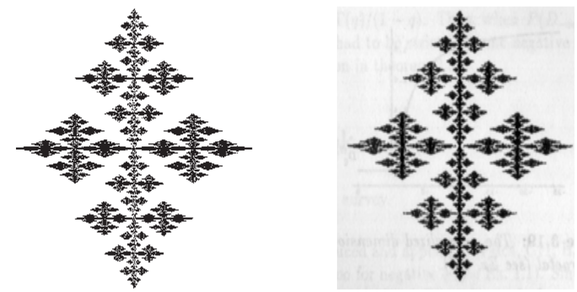

Figure 3.18: *The rosette as defined in example 3.6 on the left. To emphasize the concentration of this measure near the horizontal parts, a multifractal is provided on the right, which is based on the same IFS but with probabilities $p_i$ according to the Jacobians of the maps (i.e. $p_1 = p_3 = 1/6$, $p_2 = p_4 = 1/3$). This results in a 'homogeneous' distribution.*

with probabilities $p_i = 1/4$. Consequently the singularity exponents of $\mu^{(1)}$ are identical with the one of $\overline{\mu}^{(1)}$.
Let $\delta_n = 2^{-n}$ and take $B = [k \cdot \delta_n, (k+1)\delta_n[$ with an integer $k \in \{-2^n, \ldots, 2^n - 1\}$. These sets $B$ are the $\delta_n$-boxes with nonvanishing $\mu^{(1)}$-measure. For $0 < k < 2^n - 1$

$$\mu^{(1)}((B)_1) = \overline{\mu}([(k-1)\delta_n, (k+2)\delta_n[\times\mathbb{R}) = 3\delta_n \cdot (1-(k+1/2)\delta_n) = 3{\delta_n}^2(2^n - k - 1/2),$$

for $k = 0$: $\mu^{(1)}((B)_1) = 3{\delta_n}^2(2^n - 2/3)$ and for $k = 2^n - 1$: $\mu^{(1)}((B)_1) = 2{\delta_n}^2$. For negative $k$ replace $k$ by $|k|$. First we observe that ${\delta_n}^2 < \mu^{(1)}((B)_1) \leq 3{\delta_n}^1$. Thus the semispectra are trivial for $\alpha$ outside $[1,2]$. Omitting the three boxes corresponding to $k = -2^n, 0, 2^n - 1$ will not affect the rest of this calculation. For $\alpha \in ]1,2[$ one finds

$$\mu^{(1)}((B)_1) < {\delta_n}^\alpha \Leftrightarrow 2^n - 2^{n(2-\alpha)}/3 - 1/2 < |k| < 2^n - 1$$

which is satisfied by $2 \cdot \lfloor 2^{n(2-\alpha)}/3 - 2 \rfloor$ integers $k$. Thus

$$(F^{(1)})^-(\alpha) = \lim_{n\to\infty} \frac{\log M_{\delta_n}(\alpha)}{-\log \delta_n} = 2 - \alpha.$$

Similar $\mu^{(1)}((B)_1) \geq {\delta_n}^\alpha$ is satisfied by approximately $2^n - 2^{n(2-\alpha)}/3$ integers $k$ leading to

$$(F^{(1)})^+(\alpha) = \lim_{n\to\infty} \frac{\log N_{\delta_n}(\alpha)}{-\log \delta_n} = 1.$$

Proposition 1.10 implies

$$F^{(1)}(\alpha) = \lim_{\varepsilon\downarrow 0} \lim_{\delta\downarrow 0} \frac{\log({N^{(1)}}_\delta(\alpha+\varepsilon) - {N^{(1)}}_\delta(\alpha-\varepsilon))}{-\log\delta} = \begin{cases} 2-\alpha & \text{if } \alpha \in [1,2], \\ -\infty & \text{otherwise,} \end{cases}$$

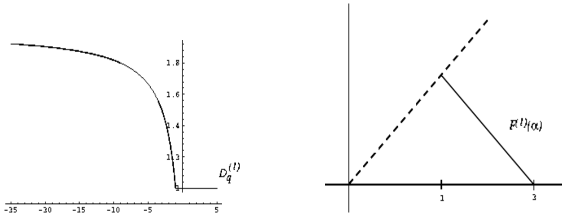

Figure 3.19: *The generalized dimensions and the spectrum of $\mu^{(1)}$ of the 'rosette' multifractal (see Ex. 3.6).*

By lemma 1.16 $T^{(1)}$ is grid-regular satisfying

$$T^{(1)}(q) = \begin{cases} 1-q & \text{if } q \geq -1, \\ -2q & \text{otherwise.} \end{cases}$$

The spectrum of $\mu^{(1)}$ is not strictly concave and so $T^{(1)}(q)$ carries not all the information about the singularities of $\mu^{(1)}$ (see Fig. 3.19).
Now let us turn to $\mu$ itself. Straightforward calculation yields

$$\gamma^{+} = \begin{cases} 3-3q+\log\left(\sqrt{h^2(q)+2^{4-3q}}-h(q)\right)/\log(1/2) & \text{if } q \geq -1, \\ 2-4q+\log\left(\sqrt{h^2(q)+2^{3-4q}}-h(q)\right)/\log(1/2) & \text{otherwise,} \end{cases}$$

where $h(q) = p^q + (1/2-p)^q$ for short. Since $\mu$ is an ordered SAMF corollary 3.2 and proposition 3.16 yield

$$T(q) = \gamma^{+} \quad (q \geq 0), \qquad T(q) \leq \gamma^{+} \quad (q < 0)$$

and $T$ is at least grid-regular for $q \geq 0$. Theorem 1.1 gives the increasing part of the spectrum $F(\alpha)$ including the maximal value (see Fig. 3.20).

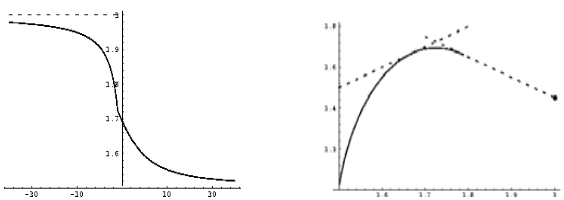

Figure 3.20: *On the left the function $\Gamma(q)/(1-q)$ and on the right its Legendre transform for the 'rosette' with $p = 1/4$. They equal the generalized dimensions for positive $q$ resp. the spectrum in the rising part.*



Finally note that (3.30) selects the maps $w_1$ and $w_3$ and that (3.32) holds. Indeed $w_1$ and $w_3$ press the measure towards the $x^{(1)}$-axis, which has dimension $1 = F(D_\infty)$. On the other hand, the construction of the invariant set $K'' = \langle w_2, w_4 \rangle$ is certainly performed only by boxes with $\mu(B) = \delta^2$. Since we expect them to be the only ones of this kind we conjecture $F(D_{-\infty}) = 1$. This value is considerably smaller than 1.45, obtained from the Legendre transform of $\Gamma(q)/(1-q)$. Thus, when $F(D_{-\infty}) = 1$ were indeed true, the upper bound $\Gamma$ of $T$ had to be strict for some negative $q$, and 'centered' would be a necessary precondition in theorem 3.3. ○

## 3.6 Conclusions. Outlook.

Our conclusions are summarized in a short survey.

- The generalized dimensions $d_q$ introduced and applied in [Falc4, Gr1, HJKPS, HP] usually take the irrelevant value $\infty$ for negative $q$ (see Ex. 1.1). Since certain multifractals develop interesting generalized dimensions only for negative $q$ (Ex. 3.5), it is important to have a new method of measuring the singularities of a multifractal.

- One essential relation in multifractal formalism is the connection of spectrum and generalized dimensions through the Legendre transformation. However, the formalism in [Falc4] requires the existence of the double limes $f(\alpha)$ for all $\alpha$. Even with the most simple multifractal (Ex. 1.1) this double limes fails to exist for large $\alpha$. To solve this problem we introduce the semispectra: they are easy to handle and have almost the same properties as the spectrum itself (Prop. 1.8, 1.10 and 1.15). Moreover, it is possible to obtain the spectrum from the singularity exponents without assuming anything about it *in advance* (theorems 1.1 and 1.2).

- As is demonstrated with most of the examples, the semispectra as well as the singularity exponents very often allow to compute the whole spectrum $F$. In particular examples, however, $F$ may carry more detailed information (Ex. 2.15 and 2.16).

- The new formalism allows to give a rigorous proof for the well known formula for the singularity exponents of self-similar measures, including negative $q$.

- The self-affine multifractals presented under the name C-SAMF reveal features of multifractal spectra different from the ones of self-similar measures. Hence they provide a greater diversity as models for objects found in nature. Moreover, we consider C-SAMFs to be important, since they allow the study of several topics such as the product of self-similar measures, local Hölder exponents and other features. See section 3.5.

- We provide multifractals with atypical spectra. They show that $F$ need not be concave and that it may possess wedges or linear parts. Furthermore, $T(q)$ is not necessarily differentiable, or can be once but not twice continuously differentiable (Ex. 2.15, 2.16, 2.17, 3.3, 3.5 and 3.6). Finally, infinite singularity exponents are found for all negative $q$ with the left-sided spectra (Ex. 2.14). For further constructed examples of this kind as well as for observations in nature see [MEH, ME, CJVP]. Exponential dimensions [GA], carefully defined in order not to reflect wrong measurement (Ex. 1.1) may turn out to be the rightly chosen formalism here.
- The interpretation of $F(\alpha)$ as the dimension of the set 'with Hölder exponent $\alpha$' is valid for certain self-similar measures (Ex. 2.10 and 2.11). For general multifractals, however, this matter is far from trivial. At least we are able to provide evidence especially for the 'most rarefied' and the 'most probable' points (section 3.5).
- As a straightforward generalization of the present work we mention the IFS consisting of contractive affinities in $\mathbb{R}^d$, which leave two given, complementary subspaces invariant and which reduce to similarities therein. Such IFS are applied to obtain fractal interpolation surfaces [GH2, BEHM].
- Finally, the present work may influence the investigations of randomly generated measures [A, Falc2, Z, Graf] and of Recurrent IFS [B, BEH, Bed2, GH2].

# Curriculum vitae

| | |
|---|---|
| 18. Juni 1961 | geboren in St.Gallen |
| 1974 - 1980 | Besuch des humanistischen Gymnasiums an der Kantonsschule St.Gallen |
| September 1980 | Maturität Typus B |
| 1980 - 1981 | Industriepraktikum und Militärdienst |
| 1981 - 1986 | Studium an der Abteilung IX für Mathematik und Physik an der ETH Zürich |
| Oktober 1986 | Diplom in Mathematik |
| 1986 - 1987 | Praktikum in einer Lebensversicherungs-Gesellschaft |
| seit April 1987 | Assistent am Departement für Mathematik der ETH Zürich und Doktorand in fraktaler Geometrie |
| März - August 1988 | Aufenthalt in den U.S.A. und in Asien |

# Dank

Eine Arbeit dieses Ausmasses ist gar nicht denkbar ohne Anregungen, ohne eine funktionierende Infrastruktur und ohne ein gesundes menschliches Umfeld.
Vielleicht wäre alles ganz anders gekommen, wäre nicht Claudia gewesen, die mir auf ihre ureigenste resolute Art den Mut für ein Studium der Mathematik einflösste. Bereits mit der Diplomarbeit führte mich mein späterer Doktorvater Prof. Blatter in die faszinierende, im Entstehen begriffene Geometrie der Fraktale ein. Im Verlaufe des vertieften Studiums zeigten sich vor allem die inspirierenden Arbeiten von Doudady et Oesterle, Falconer, Barnsley und Bedford als wegweisend. Durch die lange Zeit der Suche leitete mich Prof. Blatter mit einem Blick für das Wesentliche und von ihm stammt auch das schöne und lehrreiche Beispiel der Rosette. Erst gegen Ende dieser Arbeit fand ich die Gelegenheit zum Besuch eines internationalen Seminars, welches mir eine erste Feuerprobe für meine Ideen und vielerlei Einsichten in andere multifraktale Ansätze bescherte. In diesem Zusammenhang bin ich der SMG für ihre grosszügige Unterstützung zu Dank verpflichtet, sowie Matthias A., der mich auf die Arbeit von Schief hingewiesen hat. Letztendlich mein herzlicher Dank an Silvia für die sorgsame Durchsicht des Manuskripts.
Es scheint mir auch am Platz zu sein, hier der ETH Zürich zu danken. Die Anstellung als Assistent und die gutausgebaute Infrastruktur sind nicht wegzudenkende Voraussetzungen für die vorliegende Forschungsarbeit. Von all den vielen Kommenden und Gehenden in der Gruppe, zu der ich mich nun mittlerweile seit manchen Jahren zugehörig fühlen darf, möchte ich zwei besonders erwähnen. Da ist zum einen Nöbi, eine Art Naturkonstante in unserer Gruppe, wandelndes Orakel und Bronstein in einem. Zum anderen wäre die Arbeit mit den verschiedenen elektronischen Hilfsmitteln undenkbar ohne Beat, den Fachmann für alle Arten von 'bugs'. An sie zwei, stellvertretend für die ganze Gruppe, richte ich meinen herzlichen Dank für das wohltuende Klima im G 34.
So viele sind Stücke des Weges mit mir gegangen. Besonders am Herzen liegt mir Marie-Claire, meine Begleiterin durch viele Jahre. Mein tiefster Dank aber gehört Oliver. Sein stets waches Interesse an meiner Arbeit und sein ungebrochener Optimismus gereichten mir zu Ansporn und einem gesunden Mass Selbstkritik, dieses unabdingbare Paar Beine, auf dem jede Arbeit fortschreitet.

# Bibliography


[A] Matthias ARBEITER *Random recursive construction of self-similar fractal measures. The noncompact case.* Probab. Th. Rel. Fields: **88** (1991) pp 497–520.

[BP] R. BADII et A. POLITI *Statistical Description of Chaotic Attractors: The Dimension Function* J. Stat. Phys.: **40** (1985) pp 725–750.

[B] Christoph BANDT *Self-Similar Sets 3. Constructions with Sofic Systems* Monatsh. Math.: **108** (1989) pp 89–102.

[BG] Christoph BANDT et Siegfried GRAF *Self-Similar Sets 7. A Characterization of Self-Similar Fractals with Positive Hausdorff Measure* Proc. AMS: **114** (1992) pp 995–1001.

[Bar] Michael F. BARNSLEY *Fractals everywhere* San Diego: Academic Press 1988

[BEH] M. BARNSLEY, J. ELTON et D. HARDIN *Recurrent Iterated Function Systems* Constr. Approx.: **5** (1989) pp 3–31.

[BEHM] M. BARNSLEY, J. ELTON, D. HARDIN et P. MASSOPUST *Hidden Variable Fractal Interpolation Functions* SIAM J. Math. Anal.: **20** (1989) pp 1218–1242.

[Bed1] Tim BEDFORD *Ph. D. Thesis* Warwick Univ. (UK) (1984)

[Bed2] Tim BEDFORD *Hausdorff dimension and box dimension in self-similar sets* Proc. Conf. Topology and Measure V, Ernst-Moritz-Arndt-Univ. Greifswald (1988) pp 17–26.

[Bed3] Tim BEDFORD *Hölder Exponents and Box Dimension for Self-Affine Fractal Functions* Constr. Approx.: **5** (1989) pp 33–48.

[Bed4] Tim BEDFORD *The box dimension of self-affine graphs and repellers* Nonlinearity: **2** (1989) pp 53–71.

[BU] Tim BEDFORD et Mariusz URBANSKI *The box and Hausdorff dimension of self-similar sets* Report 88–80 Delft Univ. of Technology (1988)

[BPPV] R. BENZI, G. PALADIN, G. PARISI et A. VULPIANI *On the multifractal nature of fully developed turbulence and chaotic systems* J. Phys. A: Math. Gen.: **17** (1984) pp 3521–3531.

[Bill] P. BILLINGSLEY *Probability and Measure* New York: Wiley 1979 (Wiley Series in Probability and Statistics)

[BR] Tomas BOHR et David RAND *The entropy function for characteristic exponents* Physica D: **25** (1987) pp 387–398.

[CM] Robert CAWLEY et R. Daniel MAULDIN *Multifractal Decompositions of Moran Fractals* Advances Math.: **92** (1992) pp 196–236.

[CLP] P. COLLET, J. LEBOVITC et A. PORCIO *The Dimension Spectrum of Some Dynamical Systems* J. Stat. Phys.: **47** (1987) pp 609–644.

[CJVP] A. CRISANTI, M. H. JENSEN, A. VULPIANI et G. PALADIN *Strongly intermittent chaos and scaling in an earthquake model* Nordita preprint, Danmark (1992)

[DO] Adrien DOUADY et Joseph OESTERLE *Dimension de Hausdorff des attracteurs* C. R. Acad. Sc. Parisi, Série A: t. **290** (30 juin 1980) pp 1135–1138.

[Eggl] H. G. EGGLESTON *The fractional dimension of a set defined by decimal properties* Quart. J. of Math. (Oxford): **20** (1949) pp 31–36.

[Ell] Richard ELLIS *Large Deviations for a general Class of Random Vectors* Ann. Prob.: **12** (1984) pp 1–12.

[EM] C. J. G. EVERTSZ et B. B. MANDELBROT *Multifractal Measures* Appendix B in: 'Chaos and Fractals' by H.-O. Peitgen, H. Jürgens and D. Saupe, Springer New York (1992) pp 849–881.

[Falc1] Kenneth J. FALCONER *Geometry of Fractal sets* Cambridge Univ. Press (1985)

[Falc2] Kenneth J. FALCONER *Random fractals* Math. Proc. Cambr. Phil. Soc.: **100** (1986) pp 559–582.

[Falc3] Kenneth J. FALCONER *The Hausdorff dimension of self-affine fractals* Math. Proc. Cambr. Phil. Soc.: **103** (1988) pp 339–350.

[Falc4] Kenneth J. FALCONER *Fractal Geometry: Mathematical Foundations and Applications* John Wiley and Sons, New York (1990)

[Falc5] Kenneth J. FALCONER *The dimension of self-affine fractals II* Math. Proc. Cambr. Phil. Soc.: **111** (1992) pp 169–179.

[FM] Kenneth J. FALCONER et D. T. MARSH *Classification of quasi-circles by Hausdorff dimension* Nonlinearity: **2** (1989) pp 489–493.



[Fed] Jens FEDER *Fractals* New York, Plenum Press (1989)

[GL] Dimitrios GATZOURAS et Steven LALLEY *Hausdorff and Box dimensions of Certain Self-Affine Fractals* Indiana Univ. Math. J.: **41** (1992) pp 533–567.

[GH1] J. GERONIMO et D. HARDIN *An Exact Formula for the Measure Dimensions Associated with a Class of Piecewise Linear Maps* Constr. Approx.: **5** (1989) pp 89–98.

[GH2] J. GERONIMO et D. HARDIN *Fractal Interpolation Surfaces and a related 2-D Multiresolution Analysis* To appear in J. Math. Anal. Appl.

[GM] Steve GILBERT et Peter MASSOPUST *The exact Hausdorff Dimension for a Class of Fractal Functions* to appear J. Math. Anal. and Appl. (1992f)

[Graf] Siegfried GRAF *Statistically Self-Similar Fractals* Probab. Th. Rel. Fields: **74** (1987) pp 357–392.

[Gr1] Peter GRASSBERGER *Generalized Dimensions of strange attractors* Phys. Lett. A: **97** (1983) pp 227–230.

[Gr2] Peter GRASSBERGER *Generalizations of the Hausdorff Dimension of Fractal Measures* Phys. Lett. A: **107** (1985) pp 101–105.

[GP1] P. GRASSBERGER et I. PROCACCIA *Characterization of strange attractors* Phys. Rev. Lett.: **50** (1983) pp 346–349.

[GP2] P. GRASSBERGER et I. PROCACCIA *Measuring the strangeness of strange attractors* Physica D: **9** (1983) pp 189–208.

[GA] Neelima GUPTE et R. E. AMRITKAR *New dimensions in multifractals: The exponential dimension* Physical Review A: **39** (1989) pp 5466–5469.

[GS] R. GUTFRAIND et M. SHEINTUCH *Fractal and multifractal analysis of the sensitivity of catalytic reactions to catalyst structure* J. Chem. Phys.: **95** (1991) pp 6100–6111.

[HR] Jan HAKANSSON et Gunnar RUSSBERG *Finite size effects on the characterization of fractal sets: $f(\alpha)$ construction via box counting on a finite two-scaled Cantor set* Phys. Rev. A: **41** (1990) pp 1855–1861.

[HJKPS] T. HALSEY, M. JENSEN, L. KADANOFF, I. PROCACCIA et B. SHRAIMAN *Fractal measures and their singularities: The characterization of strange sets* Phys. Rev. A: **33** (1986) pp 1141–1151.

[HP] H. HENTSCHEL & I. PROCACCIA *The Infinite number of Generalized Dimensions of Fractals and Strange Attractors* Physica D: **8** (1983) pp 435–444.

[Hut] John HUTCHINSON *Fractals and Self Similarity* Indiana Univ. Math. J.: **30** (1981) pp 713–747.

[JKL] M. JENSEN, L. KADANOFF et A. LIBCHABER *Global Universality at the Onset of Chaos: ...* Phys. Rev. Lett.: **55** (1985) pp 2798–2801.

[JKP] M. JENSEN, L. KADANOFF et I. PROCACCIA *Scaling structure and thermodynamics of strange sets* Phys. Rev. A: **36** (1987) pp 1409–1420.

[Koh] Mahito KOHMOTO *Singularities in the Thermodynamic Formalism of Multifractals* J. Phys. Soc. Japan: **60** (1991) pp 2876–2880.

[K] Norio KONO *On Self-Affine Functions* Japan J. Appl. Math.: **3** (1986) pp 259–269.

[Lan] Oscar E. LANFORD III *Entropy and Equilibrium States in Classical Statistical Mechanics* in: Statistical Mechanics and Mathematical Problems, Springer Lecture Notes in Physics: **20** (1986) pp 1–113.

[L] Larry LIEBOVITCH et Tibor TOTH *A fast algorithm to determine fractal dimensions by box counting* Physics Letters A: **141** 1989 pp 386–390.

[Lop] Artur LOPES *The Dimension Spectrum of the Maximal Measure* SIAM J. Math. Anal.: **20** 1989 pp 1243–1254.

[MEH] B. B. MANDELBROT, C. J. G. EVERTSZ et Y. HAYAKAWA *Exactly self-similar left-sided multifractal measures* Physical Review A: **42** 8 (1990) pp 4528–4536.

[ME] B. B. MANDELBROT et C. J. G. EVERTSZ *Exactly self-similar multifractals with left-sided* $f(\alpha)$ To appear in: Fractals and Disordered Systems, Editors A. Bunde and S. Havlin

[Ma1] Peter MASSOPUST *Space Curves generated by Iterated Function Systems* Thesis, Georgia Institute of Technology July 1986.

[Ma2] Peter MASSOPUST *Vector-Valued Fractal Interpolation functions and their Box dimension* to appear Aequat. Math. (1992f)

[Ma3] Peter MASSOPUST *The Box Dimension for a Class of Vector-Valued Hidden Variable Fractal Functions* Preprint Vanderbilt Univ., Nashville, Tennessee (1991)

[Mor] P. A. P. MORAN *Additive Functions of intervals and Hausdorff Measure* Proc. Cambr. Phil. Soc.: **42** (1946) pp 15–23.

[Mu] Curt MC MULLEN *The Hausdorff Dimension of general Sierpiński Carpets* Nagoya Math. J.: **96** (1984) pp 1–9.



[OWY] E. OTT, W. WITHERS et J. YORKE *Is the Dimension of Chaotic Attractors invariant under Coordinate Changes?* J. Stat. Phys.: **36** (1984) pp 687–697.

[Pth] K. R. PARTHASARATHY *Probability Measures on Metric Spaces* in: Probability and Math. Statistics, Academic Press New York 1967

[R] D. A. RAND *The singularity spectrum for hyperbolic Cantor sets and attractors* Preprint Warwick Univ. (1986)

[Rog] C. A. ROGERS *Hausdorff Measures* Cambridge Univ. Press (1970)

[Sa] Enn SAAR et Veikko SAAR *Large-scale Patterns* Manuscript (Cambridge, Institute of Astronomy)

[Sch] Andreas SCHIEF *SOSC and OSC are Equivalent* Preprint Department of Mathematics Ludwig-Maximilians-Universität Munich

[S] Jörg SCHMELING et Rainer SIEGMUND-SCHULTZE *The singularity spectrum of self-affine fractals with a Bernoulli measure* Manuscript (Berlin)

[Tél] Tamás TEL *Fractals, Multifractals and Thermodynamics* Z. Naturforsch. A: **43** (1988) pp 1154–1174.

[TV] Tamás TEL et Tamás VICSEK *Geometrical multifractality of growing structures* J. Phys. A: Math. Gen.: **20** (1987) L835–L840.

[Tr] Claude TRICOT *Two definitions of fractal dimension* Math. Proc. Cambr. Phil. Soc.: **91** (1982) pp 57–74.

[U] Mariusz URBANSKI *The Probability Distribution and Hausdorff Dimension of Self-affine Functions* Probab. Th. Rel. Fields: **84** (1990) pp 377–391.

[V1] Tamás VICSEK *Fractal Growth Phenomena* World Scientific, Singapore, 1989

[V2] Tamás VICSEK *Fractal models for diffusion controlled aggregation* J. Phys. A: Math. Gen.: **16** (1983) L647–L652.

[W] Helmut WEGMANN *Die Hausdorff Dimension von kartesischen Produkten metrischer Räume* J. Reine u. Angewandte Math.: **246** (1969) pp 46–75.

[Y] Lai-Sang YOUNG *Dimension, Entropy and Lyapunov Exponents* Ergod. Th. & Dynam. Sys.: **2** (1982) pp 109–124.

[Z] U. ZAEHLE *Self-similar Random Measures II. A Generalization to Self-affine Measures.* Math. Nachr.: **146** (1990) pp 85–98.

# Index

**Citations**