A strong Tauberian theorem for characteristic functions

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Abstract

Using wavelet analysis we show that if the characteristic function of a random variable X can be approximated at 0 by some polynomial of even degree $2p$ then the moment of order $2p$ of X exists. This strengthens a Tauberian-type result by Ramachandran and implies that the characteristic function is actually $2p$ times differentiable at 0. This fact also provides the theoretical basis for a wavelet based non-parametric estimator of the tail index of a distribution.

Introduction

When studying the existence of moments $\beta_q = \mathbb{E}|X|^q$ of a random variable X it is essential to understand the behavior of its characteristic function $\phi(u) = \mathbb{E}[\exp(iuX)]$ at 0, in particular its degree of differentiability, and more precisely its Hölder regularity.

A function g is called *Hölder-regular* of degree $r \geq 0$ at u_0, denoted $g \in \mathcal{H}^r(u_0)$, if there exists a polynomial P of degree at most $\lfloor r \rfloor := \max\{n \in \mathbb{N}_0 : n \leq r\}$ such that

$$|g(u) - P(u)| \leq C \cdot |u - u_0|^r \tag{1}$$

If g is actually $\lfloor r \rfloor$ times differentiable, then P must be equal to the Taylor polynomial and we call g *Taylor regular* of degree r. Functions which obey (1) but are not $\lfloor r \rfloor$ times differentiable are abundant. An example is $g(u) = 1+u+u^2+u^{3.1}\sin(1/u)$ which is only once differentiable at $u = 0$ but Hölder regular for any $r \leq 3.1$.

However, differentiability of a characteristic function at 0 is intimately related to the existence of moments: for any positive integer p we have $\phi^{(2p)}(0) = (-1)^p\mathbb{E}[X^{2p}]$ whenever one of the two is defined. To make this point even stronger, Ramachandran shows: if $\phi(u)$ is symmetric and Taylor regular of degree $r = 2p + \delta$ $(0 < \delta < 2)$ at 0 then all moments β_q of degree $q < r$ are finite (see [6, Thm. 3]).

Thereby, the assumption of ϕ being symmetric is not required (see, e.g., Lukacs [5, Thm. 2.2.2.]). Indeed, the symmetric function $\operatorname{Re}\phi(u)$ is the characteristic function of the random variable Y which is equal to X with probability 1/2, and equal to $-X$ with probability 1/2. Since $|Y| = |X|$ a.s. we have $\mathbb{E}|Y|^q = \mathbb{E}|X|^q = \beta_q$ and it suffices to consider $\operatorname{Re}\phi$.

The main result

We are able to strengthen Ramachandran's result considerably via the following:

Theorem 1 *Let $r > 0$ be positive and p integer such that $2p \le r < 2p+2$. If $\mathrm{Re}\,\phi(u) \in \mathcal{H}^r(0)$ then $\phi(u)$ is $2p$-times differentiable at 0.*

Proof

For $p = 0$ there is nothing to show. For $p \ge 1$ denote by ψ_p the function with real valued, non-negative Fourier transform

$$\Psi_p(\nu) = \nu^{2p} \exp\left(-\nu^2/4\right). \tag{2}$$

Up to a constant factor, $\psi_p(u)$ is the $2p$-th derivative of $\exp(-u^2)$.

Now consider the wavelet coefficients of $\mathrm{Re}\,\phi$ of scale s at t which are given by

$$W_p(s,t) := \int \tfrac{1}{s}\psi_p(\tfrac{u-t}{s})\mathrm{Re}\,\phi(u)du. \tag{3}$$

The Fourier transform of $\frac{1}{s}\psi_p(\frac{u}{s})$ being $\Psi_p(s\nu)$, Parseval's identity yields

$$W_p(s,0) = \mathrm{Re}\int \tfrac{1}{s}\psi_p(\tfrac{u}{s})\phi(u)du = \mathrm{Re}\int \Psi_p(s\nu)dF_X(\nu) = \mathbb{E}[\Psi_p(sX)]. \tag{4}$$

Monotone convergence gives

$$\sup_{s>0}\frac{W_p(s,0)}{s^{2p}} = \sup_{s>0}\mathbb{E}[X^{2p}\exp(-s^2X^2/4)] = \mathbb{E}[X^{2p}] \in \mathbb{R}_0^+ \cup \{\infty\}.$$

To control the decay of $W_p(s,0)$ at 0 note that $\mathrm{Re}\,\phi$ is Hölder regular of degree $2p$ which can be seen by absorbing higher order terms into $C|u|^{2p}$ from (1) for $|u| \le 1$ and by using $|\phi(u)| \le 1$ for $|u| > 1$. Also, $\mathrm{Re}\,\phi$ is even. Thus, there is an even polynomial Q of degree at most $2p-2$ and a constant C_1 such that $|\mathrm{Re}\,\phi(u) - Q(u)| \le C_1|u|^{2p}$. By repeated integration by parts, we find that $\int Q(u)\psi_p(u/s)du = 0$ for any $s > 0$. From this, we find

$$\begin{aligned} |W_p(s,0)| &= \left|\int \tfrac{1}{s}\psi_p(\tfrac{u}{s})(\mathrm{Re}\,\phi(u) - Q(u))du\right| \le C_1 \int \tfrac{1}{s}\left|\psi_p(\tfrac{u}{s})\right| |u|^{2p}du \\ &= C_1|s|^{2p}\int |\psi_p(v)|\,|v|^{2p}dv = C_2|s|^{2p}. \end{aligned} \tag{5}$$

Note that $C_2 < \infty$ due to the exponential tail of ψ_p. The claim follows. ♢

Theorem 1 strengthens Ramachandran [6, Thm. 3] as follows.

Theorem 2 *Let $r > 0$. If $\mathrm{Re}\,\phi \in \mathcal{H}^r(0)$ then $\beta_q < \infty$ for $0 \le q < r$.*

Remark 1: The stable laws with $\phi(u) = \exp(-|u|^r)$ $(0 < r < 2)$ demonstrate that β_r may not be finite in general. However, if r is an even integer, we always have $\beta_r < \infty$.

Proof

In the light of Theorem 1 we may simply quote Ramachandran [6, Thm. 3] to complete the proof. However, just a little more work provides an alternative proof to this classical result. In the notation of (1) and (2) we have $\int P(u)\psi_{p+1}(u/s)du = 0$ ($p \geq 0$). Similar as in (5) we get

$$\mathbb{E}[\Psi_{p+1}(sX)] \leq C \int \tfrac{1}{s} \left|\psi_{p+1}(\tfrac{u}{s})\right| |u|^r du = C_3|s|^r. \tag{6}$$

Using $\Psi_{p+1}(x) \geq e^{-1/4} > 1/2$ for $1 \leq |x| \leq 2$ ($p \geq 0$) we find

$$\int_{2^k \leq |x| \leq 2^{k+1}} dF_X(x) = \mathbb{E}[\mathbb{1}_{[-2,-1]\cup[1,2]}(2^{-k}X)] \leq 2\mathbb{E}[\Psi_{p+1}(2^{-k}X)] \leq 2C_3 \cdot 2^{-kr}.$$

It follows immediately that the moments β_q are finite for $0 \leq q < r$. $\diamondsuit$

Classical results
Theorem 1 is quite well known in the case $r = 2$:

Theorem 3 *If $|\phi(u) - 1| \leq C \cdot u^2$, then the variance of the law is finite.*

Proof
Note that $|\mathrm{Re}\,\phi(u) - 1| \leq C \cdot u^2$. But $1 - \mathrm{Re}\,\phi(u) = \mathbb{E}[2\sin^2(uX/2)]$. Since $X^2 = \lim_{u\to 0} \sin^2(uX)/u^2$ a.s. it suffices then to apply Fatou's lemma. $\diamondsuit$

Alternative choice of wavelet:
Our choice of wavelet in the proofs of Theorem 1 and Theorem 2 (see (2)) is by far not the only possible. Here is one interesting alternative. Motivated by the classical argument of Theorem 3 we may replace Ψ_p by $\sin^{2p}$ which leads to a somewhat more elementary computation akin to the proofs of Ramachandran and Lukacs.
Indeed, using $(\sin sx)^{2p} = (2i)^{-2p}(\exp(isx) - \exp(-isx))^{2p}$ we obtain (cpre. (4))

$$\mathbb{E}[\sin^{2p}(sX)] = (-1/4)^p \sum_{k=0}^{2p} \binom{2p}{k} (-1)^k \phi(s(2p-2k)) =: \Delta_p^s \phi(0).$$

Here, we introduced an order $2p$ difference operator Δ_p^s centered at 0 which is in essence the $2p$-fold convolution of the Haar wavelet. Being non-negative real $\Delta_p^s\phi(0)$ equals $\Delta_p^s \mathrm{Re}\,\phi(0)$. Also, Δ_p^s possesses the appropriate number of vanishing moments, meaning that $\Delta_p^s Q = 0$ for Q from (5), since Q is of degree at most $2p-2$ and $\sum_{k=0}^{2p} \binom{2p}{k}(-1)^k k^m = 0$ for any $0 \leq m \leq 2p-2$. Setting C_1 as in (5) we find

$$\mathbb{E}[\sin^{2p}(sX)] = |\Delta_p^s(\mathrm{Re}\,\phi - Q)| \leq 4^{-p} \sum_{k=0}^{2p} \binom{2p}{k} C_1 |2p-2k|^{2p} |s|^{2p}.$$

Using Fatou as in Theorem 3 establishes finiteness of β_{2p} and, therefore, Theorem 1. To establish Theorem 2 in this setup, write $\mathbb{E}[\sin^{2p+2}(sX)] = |\Delta_{p+1}^s(\mathrm{Re}\,\phi - P)| \leq C_4|s|^r$ and use $\sin\theta \geq \sqrt{3}/2$ for $\pi/3 \leq \theta \leq 2\pi/3$ to establish finiteness of β_q for $0 \leq q < r$ as in Theorem 2 above. $\diamondsuit$

Remark 2: Lukacs [5] uses essentially the same operator $\Delta_p^s\phi(t)$, centered at an arbitrary t; he also uses the vanishing moments. But writing $\Delta_p^s\phi(t) = \Delta_p^s P(t) +$

$\Delta_p^s(\phi - P)(t)$ his estimates contain a term s^{2p} in addition to s^r. Lukacs assumes differentiability of ϕ in order to deal with the extra term s^{2p}. This crucial fact is obscured by a minor typing error in the statement of the theorem which makes it appear as if Lukacs assumed only Hölder regularity.

Further regularity results

The alternative wavelets mentioned above provide regularity conditions that do not require an approximating polynomial of the characteristic function:

Corollary 4 *Let r be positive real and m and n positive integers. Then*

$$|\Delta_m^s \phi(0)| \leq C|s|^{2m} (s \to 0^+) \quad \Rightarrow \quad \phi^{(2m)}(u) \text{ exists.}$$
$$|\Delta_n^s \phi(0)| \leq C|s|^r \quad (s \to 0^+) \quad \Rightarrow \quad \beta_q < \infty \ (0 \leq q < r).$$

In the special case $n = 1$ we are lead to the following notion: A function g is said to lie in the *Zygmund class* $\dot{\mathcal{C}}^1(u_0)$ if g satisfies

$$|g(u_0 + s) - 2g(u_0) + g(u_0 - s)| \leq C|s|^r \tag{7}$$

with $r = 1$. The condition (7) is equivalent to $g \in \mathcal{H}^r(u_0)$ for $0 < r < 1$, but the inclusion $\mathcal{H}^1(u_0) \subset \dot{\mathcal{C}}^1(u_0)$ is strict. However, since $1 - \operatorname{Re}\phi(2s) = 2\Delta_1^s\phi(0)$ the real part of any characteristic function in Zygmund class $\dot{\mathcal{C}}^1(0)$ lies actually in $\mathcal{H}^1(0)$.
We study the derivatives of characteristic functions in terms of the following spaces: we set $\dot{\mathcal{C}}^r(u_0) = \mathcal{H}^r(u_0)$ for $0 < r < 1$; for $r > 1$ we denote by $\dot{\mathcal{C}}^r(u_0)$ the space of functions who's derivatives exist around u_0 up to order m ($m < r \leq m + 1$) and who's m-th derivative lies in $\dot{\mathcal{C}}^{r-m}(u_0)$.
For general functions, control on the regularity of derivatives is not obvious (see the example of a function $g \in \mathcal{H}^{3.1}(0)$ from the introduction). In order to gain such control we make use of the so-called *two-microlocalization*: with reference to (3) the function $\operatorname{Re}\phi(u)$ is said to belong to $\mathcal{C}^{r,r'}(u_0)$ if

$$|W_{p+2}(s,t)| \leq Cs^r \left(1 + \frac{|t - u_0|}{s}\right)^{-r'}. \tag{8}$$

Here, the wavelet ψ_{p+2} needs to be sufficiently regular, i.e., $2p + 4 > r + 2$.
We will need the following results

$$\begin{array}{rcll} \mathcal{H}^r(u_0) & \subset & \mathcal{C}^{r,-r}(u_0) & (r > 0) \\ \mathcal{C}^{r,r'}(u_0) & \subset & \mathcal{H}^r(u_0) & (r' > -r,\ r > 0 \text{ non-integer}) \\ \mathcal{C}^{1,0}(u_0) & \subset & \dot{\mathcal{C}}^1(u_0) & \end{array} \tag{9}$$

The first two inclusions are found in [3, Prop. 1] as well as [4, Prop. 1.3]. For the third inclusion one proves the somewhat more general fact that $g \in \mathcal{C}^{r,0}(u_0)$ ($1 \leq r < 2$) implies (7) by applying the mean value theorem twice in [3, p. 292].
At this point, the wavelet-maxima-property of characteristic functions proves useful:

Lemma 5 *The wavelet coefficients (3) of ϕ and $\operatorname{Re}\phi$ are both maximal at 0:*

$$|W_p^{\operatorname{Re}\phi}(s,t)| \leq |W_p^{\phi}(s,t)| \leq W_p(s,0) = \mathbb{E}[\Psi_p(sX)].$$

Proof
This follows easily from Parseval's identity, i.e.,

$$W_p^\phi(s,t) = \int \tfrac{1}{s}\psi_p(\tfrac{(u-t)}{s})\phi(u)du = \int \Psi_p(s\nu)\exp(-it\nu)dF_X(\nu), \tag{10}$$

combined with $\Psi_p \geq 0$, and (4). ♢

Note that $\mathrm{Re}\,\phi \in \mathcal{H}^r(0)$ implies $|W_{p+2}(s,0)| \leq C|s|^r$ due to (9) and (8) with $t = u_0$. On the other hand, this bound implies with Lemma 5 that $\phi \in \mathcal{C}^{r,0}(u_0)$ for any u_0. Notably, the same constant C can be used in the wavelet bound (8) for all u_0. This fact is indicated by writing $\phi \in \mathcal{C}^{r,0}(\mathbb{R})$ and analogously for other regularity spaces. Now, (9) implies

Corollary 6 *For non-integer $r > 0$ we have*

$$\mathrm{Re}\,\phi \in \mathcal{H}^r(0) \quad\Leftrightarrow\quad |W_{p+2}(s,0)| \leq Cs^r \ (s>0) \quad\Leftrightarrow\quad \phi \in \mathcal{H}^r(\mathbb{R}) \quad\Leftrightarrow\quad \mathrm{Re}\,\phi \in \mathcal{H}^r(\mathbb{R}).$$

Due to Theorem 1 and classical results, $\mathrm{Re}\,\phi \in \mathcal{H}^r(0)$ implies that the derivatives of ϕ exist and are uniformly continuous up to order m ($m < r \leq m+1$). It is known that [2, 4]

$$\mathrm{Re}\,\phi(u) \in \mathcal{C}^{r,r'}(u_0) \quad\Rightarrow\quad \frac{\partial^m}{\partial u^m}\mathrm{Re}\,\phi(u) \in \mathcal{C}^{r-m,r'}(u_0)$$

This precise control of the regularity of the derivatives via two-microlocalization can be easily verified and sharpened for even order derivatives of characteristic functions:

Lemma 7 *Assume $\beta_{2m} < \infty$ and $p > m$. Then (analogous for $\mathrm{Re}\,\phi$)*

$$W_{p-m}^{\phi^{(2m)}}(s,t) = (-1)^m s^{-2m} \cdot W_p^\phi(s,t).$$

Proof
Consider the probability distribution $dF_m(x) = \frac{1}{\beta_{2m}}x^{2m}dF_X(x)$ with characteristic function

$$\phi_m(u) = \frac{1}{\beta_{2m}}\int e^{iux}x^{2m}dF_X(x) = \frac{1}{\beta_{2m}}\mathbb{E}[e^{iuX}X^{2m}] = \frac{(-1)^m}{\beta_{2m}}\phi^{(2m)}(u). \tag{11}$$

A direct computation using Parseval's identity (10) leads to the claimed relation. ♢

In particular, $\mathrm{Re}\,\phi^{(2m)}$ and $\phi^{(2m)}$ are, up to a factor, characteristic functions.

Corollary 8 *For all $r > 0$ we have:* $\mathrm{Re}\,\phi \in \mathcal{H}^r(0) \quad\Rightarrow\quad \phi \in \dot{\mathcal{C}}^r(\mathbb{R})$.
For $r = 2p$ even, $\phi^{(2p)}$ exists and is uniformly continuous.
For $r = 2p+1$ odd, $\phi^{(2p)} \in \dot{\mathcal{C}}^1(\mathbb{R})$ and $\mathrm{Re}\,\phi^{(2p)} \in \mathcal{H}^1(\mathbb{R})$.

Remark 3: The Cauchy distribution with $\phi(u) = \exp(-|u|)$ lies in $\mathcal{H}^1(0)$ but is not differentiable. A condition for $\phi^{(2p)}$ being differentiable at 0 and, thus, everywhere is provided in [2, Prop. 1] and [4, Prop 1.5].

An alternative approach to using wavelets is a Littlewood-Paley decomposition of $\mathrm{Re}\,\phi$ into functions $\Delta_j \mathrm{Re}\,\phi$. To allow recycling of notation, consider $\Psi_0(\nu) = h(\nu/2) - h(\nu)$ where h is a function of the Schwartz class such that $0 \le h \le 1$, $h(\nu) = 1$ for $|\nu| \le 1/2$ and $h(\nu) = 0$ for $|\nu| \ge 1$. Then, by definition, the Fourier transform of $\Delta_j \mathrm{Re}\,\phi$ equals $\Psi_0(2^{-j}\nu)$ times the Fourier transform of $\mathrm{Re}\,\phi$.
In the notation of (3) we get $\Delta_j \mathrm{Re}\,\phi(u) = W_0(2^{-j}, u)$. Since Ψ_0 is positive, Lemma 5 holds also in this context; with (4), (6) and $\Psi_0 \le K\Psi_p$ for some $K > 0$ we see that $\mathrm{Re}\,\phi \in \mathcal{H}^r(0)$ implies

$$|\Delta_j \mathrm{Re}\,\phi(u)| \le \Delta_j \mathrm{Re}\,\phi(0) = \mathbb{E}[\Psi_0(2^{-j}X)] \le KC_3 2^{-jr}$$

These estimates can be used in a manner similar to the arguments provided earlier.

Statistical estimation
As an application we mention a wavelet-based non-parametric estimator of the *tail-exponent*

$$\lambda := \sup\{q > 0 \,:\, \text{the moment } \beta_q \text{ is finite}\}$$

of the distribution of X. As is well known $\beta_q < \infty$ $(q > 0)$ implies $\mathrm{Re}\,\phi \in \mathcal{H}^q(0)$ (see [5, Thm. 2.2.1.]). With Theorem 2 it follows that $\lambda = \sup\{r > 0 \,:\, \mathrm{Re}\,\phi \in \mathcal{H}^r(0)\}$. With Corollary 6 we conclude

Corollary 9 *Provided $2n > \lambda + 2$, the tail exponent λ satisfies*

$$\lambda \quad = \quad \sup\{r > 0 \,:\, \mathbb{E}[\Psi_n(sX)] \cdot s^{-r} \text{ is bounded for } s \to 0^+\}$$

In conclusion, the estimation of λ can be based on estimating $\mathbb{E}[\Psi_n(sX)]$ for various values of $s > 0$ followed by a least square linear regression in log-log towards fitting $\mathbb{E}[\Psi_n(sX)] \simeq s^\lambda$. Various values of n should be employed to ensure $2n > \lambda$. The exact setup and performance of this estimation procedure is detailed in [1].

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References

[1] P. Gonçalves and R. Riedi. Diverging moments and parameter estimation. *J. Amer. Stat. Assoc.*, 100 (472):1382–1393, December 2005.

[2] S. Jaffard. Pointwise smoothness, two-microlocalization and wavelet coefficients. *Publicacions Matematiques*, 35:155–168, 1991.

[3] S. Jaffard. Local behavior of Riemann's function. *Contemporary Mathematics*, 189:287–307, 1995.

[4] S. Jaffard and Y. Meyer. *Wavelet Methods for Pointwise Regularity and Local Oscillations of Functions*, volume 123 (587). American Mathematical Society, Sept. 1996.

[5] E. Lukacs. *Developments in Characteristic Function Theory.* Oxford University Press, New York, macmillan co. edition, 1983.

[6] B. Ramachandran. On characteristic functions and moments. *Sankhya, Indian J. of Statistics, Series A*, 31:1–12, March 1969.