On Non-Scale-Invariant Infinitely Divisible Cascades

Pierre Chainais, Rudolf Riedi, and Patrice Abry

***Abstract*—Multiplicative processes, multifractals, and more recently also infinitely divisible cascades have seen increased popularity in a host of applications requiring versatile multiscale models, ranging from hydrodynamic turbulence to computer network traffic, from image processing to economics. The methodologies prevalent as of today rely to a large extent on iterative schemes used to produce infinite detail and repetitive structure across scales. While appealing, due to their simplicity, these constructions have limited applicability as they lead by default to power-law progression of moments through scales, to non-stationary increments and often to inherent log-periodic scaling which favors an exponential set of scales. This paper studies and develops a wide class of infinitely divisible cascades (IDC), thereby establishing the first reported cases of controllable scaling of moments in non-power-law form. Embedded in the framework of IDC, these processes exhibit stationary increments and scaling over a continuous range of scales. Criteria for convergence, further statistical properties, as well as MATLAB routines are provided.**

***Index Terms*—Fractional Brownian motion, infinitely divisible cascades (IDC), multifractal, multiplicative cascades, multiscaling, random walk, turbulence.**

I. INTRODUCTION

SCALING behavior has become a welcome parsimonious description of complexity in a host of fields including natural phenomena such as turbulence in hydrodynamics, human heart rhythm in biology, spatial repartition of faults in geology, as well as mankind activities such as traffic in computer networks and financial markets. The multifractal formalism (see [1] for an extensive set of original references) has received much attention as one of the most popular framework to describe and analyze signals and processes that exhibit scaling properties, covering and connecting both local scaling and global scaling in terms of sample moments.

The term *scale invariance*, e.g., refers in various fields to a relation between the absolute moments of increments $\delta_\tau X(t) = X(t+\tau) - X(t)$ of a process X and the *lag* τ in form of a power law. More precisely, scale invariance is then described by a set of multifractal exponents $\zeta(q)$ defined through

$$\mathbb{E}|\delta_\tau X(t)|^q = C_q(\tau)\tau^{\zeta(q)}, \qquad \text{as } \tau \to 0 \tag{1}$$

where $C_q(\tau)$ is assumed either to be constant, to be bounded between positive constants, or to be a more general function[1] depending on the context. For instance, statistically self-similar processes such as fractional Brownian motion $B_H(t)$ [2] with Hurst exponent H fit into this framework with $\zeta(q) = qH$ and C_q constant equal to $\mathbb{E}[|B_H(1)|^q]$. The binomial multiplicative cascades, among others, fit with a strictly convex function $\zeta(q)$ and bounded $C_q(\cdot)$. The multifractal formalism connects the scaling exponents $\zeta(q)$ via the Legendre transform to the local degree of regularity of the path of the process.

In real world applications, the notion of $\zeta(q)$ in (1) is of limited use, since one is able to observe only a limited range of scales from actual data. For clarity, when scaling laws are meant to hold for scales or lags $\tau_{\min} \le \tau \le \tau_{\max}$ we use the term *multiscaling*. Note that for the multifractal formalism to apply rigorously one needs scaling as in (1) down to infinitely small scales.

In addition, the functional form of a power law in (1) can be limiting in applications, such as in networking [3], [4]. The framework of the infinitely divisible cascades (IDC), introduced first as a concept of analysis in fluid turbulence (see [3], [5]–[9]), answers to both shortcomings. By integrating the contribution of all scales in a range of interest, IDC analysis allows for more flexible scaling and thus better fitting of data by setting

$$\mathbb{E}|\delta_\tau X(t)|^q = C_q \exp[-\zeta(q)n(\tau)], \quad \text{for } \tau_{\min} \le \tau \le \tau_{\max} \tag{2}$$

where the function $n(\tau)$ is assumed monotonous and can be interpreted as the *depth* of the cascade. Such a behavior is analyzed in terms of a *cascading mechanism* through the scales from $\tau_{\max}$ to $\tau_{\min}$. Moreover, the IDC framework (2) encompasses the scale invariance (1) as the special case $n(\tau) = -\log\tau$.

Besides the broader context of the IDC framework in terms of scaling laws and ranges, a further difference to multifractal analysis may be noted in its spirit. Multifractal theory uses notions such as the scaling exponents $\zeta(q)$ which tend to be defined as to exist *a priori* and not to put any condition on the analyzed process (compare footnote 1) and it is concerned with inferring fine-grained, local properties of processes and signals from global scaling in various settings (see [1] for an extensive set of original references).

The framework of IDC, on the other hand, formulates a condition on the process or time series at hand, namely, *separability* of $\mathbb{E}|\delta_\tau X(t)|^q$ as a function of scale and order according to (2). This functional form may or may not hold for a process and therefore provides a true property of a process beyond a statistical description of the kind of scale invariance (1).

[1]Multifractal analysis usually works with a definition which applies to any process and which reads as: $\liminf_{\tau\to 0} \log_\tau C_q(\tau) = 0$.

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Note that both multifractal analysis and IDC scaling can be formulated in terms of wavelet coefficients by replacing increments (1) by wavelet coefficients[2] (see [1], [3], [8], [10]–[13] and references therein for original developments, applications, and surveys).

While analysis tools for multiscaling processes and infinitely divisible cascades have been widely developed (see [3], [5]–[9]), only few recent works proposed actual models and tools for synthesis of processes with prescribed and controllable IDC scaling. Since the *binomial cascades* popularized by Mandelbrot [14]–[16], multiplicative cascades have always played a central role as a paradigm of multifractals, leading to advances on random self-similar measures of considerable generality.

Shortly after the turn of the millennium, the time seemed ripe for IDC processes. In a very original approach, Schmitt and Marsan [17] describe scale-invariant infinitely divisible cascades by a stochastic equation resulting from the densification of a discrete multiplicative cascade. Their work gives decisive indications toward the unification between Mandelbrot's approach and the infinitely divisible cascades approach; however, it does not cover scaling properties in details. Barral and Mandelbrot [18] introduced the *multifractal products of cylindrical pulses* (MPCP) also called *compound Poisson cascades* (CPC) and provided their rigorous multifractal description. While cast as multiplicative cascades in [18], the CPC show infinitely divisible power-law scaling and are but a special case of IDC processes mentioned later. By prescribing the correlation function of the increments $\delta_\tau X$ of a random walk, Bacry, Delour, and Muzy [19], [20] introduced the pioneering *multifractal random walk* (MRW) that later turned out to be a particular case of a more general framework [21], [22] (see below). Finally, Bacry and Muzy [21], [23] introduced *log-infinitely divisible multifractal processes* and provided strong results on convergence and scaling behavior, extending some of the results for CPCs [18].

Inspired by earlier ideas of Mandelbrot [24] but independently from the above, *infinitely divisible cascading processes* were introduced in [22], [25], with the main goal of providing processes with controllable non-power-law scaling, especially in the framework of CPC with their associated random walks. This paper follows up with rigorous results and practical algorithms on compound Poisson and log-normal cascades as special cases of IDC processes. Doing so, we introduce the non-scale-invariant *infinitely divisible cascading* (IDC) noise, motion, and random walk. These continuous-time continuous-scale processes possess stationary increments and exhibit prescribed departures from power-law behaviors in the sense that $n(\tau) \neq -\log \tau$ in (2).

In Section II, we recall the basic definition and properties of the IDC noise and point out its interesting degrees of freedom. Doing so, we provide straightforward extensions of recent convergence results [26]. In Sections III and IV, we introduce the IDC motion and their associated *random walk* (IDC random walk) and study their statistical properties. For both the IDC motion and random walk, we put the emphasis on pinpointing their departures from power-law behaviors as accurately as possible. In Section V, we provide numerical simulations of non-scale-invariant processes; in Section VI, we give details on practical algorithms for IDC processes simulation. Conclusions and perspectives are reported in Section VII. For the sake of flow, we postpone mathematical complements and proofs to the Appendices I–V. Practical properties of these processes relevant for applications are detailed in a companion paper [27].

II. Infinitely Divisible Cascading Noise

A. Background

The distinguishing and defining common feature of cascades consists in an underlying multiplicative construction which is iterated across scales. The well-known canonical binomial cascade as introduced by Mandelbrot [14], [15] may be viewed as the archetype of multifractal random measures. Among the several ways of defining the binomial cascade, the most useful in our context is via the iterative products

$$\beta_n(t) = \prod_{\{(j,k):1\le j\le n, t\in I_{j,k}\}} W_{j,k} = \beta_{n-1}(t) \prod_{\{k:t\in I_{n,k}\}} W_{n,k}. \tag{3}$$

Here, $I_{j,k}$ stands for the nested dyadic intervals $[k2^{-j},(k+1)2^{-j})$ and $W_{j,k}$ denotes independent and identically distributed (i.i.d.) positive random variables of mean one ($\mathbb{E}W_{j,k} = 1$). By construction, β_n is constant over each interval $I_{n,k}$.

An equivalent construction of the binomial cascade emphasizes the measure-theoretic aspect by considering the β_n as densities and studying their distribution functions $X_n(t) = \int_0^t \beta_n(s)ds$. As positive martingales, these converge weakly to a limiting distribution X [16], [28], which exhibits scaling of the form of (1) with bounded $C_q(\cdot)$. Attractive from a signal processing point of view is the iterative aspect of (3) which allows for fast, tree-based synthesis algorithms as the one used for the so-called multifractal wavelet model (MWM model) [29]. This underlying tree structure is inherited from the nested arrangement of the $I_{j,k}$ which may be represented by the points $((k+1/2)2^{-j}, 2^{-j})$ in the (time, scale)-plane (see Fig. 1, left).

However, such cascades have two major drawbacks. They are *not strictly stationary* since the construction is not time-shift invariant; this may result in "blocking effects." Further, by construction, the scaling of moments is log-periodic and favors, in particular, the scale ratio equal to 2.

Following a more recent idea of Mandelbrot [24], one may overcome both drawbacks by replacing the rigid, nested arrangement of multipliers $W_{j,k}$ of the binomial cascade by a planar *marked Poisson point process* $\{(t_i, r_i, W_i)\}_i$; this lead to the MPCP, also called CPC (see Fig. 1, center). More precisely, introducing the cone

$$\mathcal{C}_r(t) = \{(t', r') : r \le r' \le 1, t - r'/2 \le t' < t + r'/2\}$$

[2]However, as far as non-scale-invariant objects are concerned, one has to take care. Indeed, the fact that scale-invariance exponents (1) do not depend on the wavelet base is deeply related to the scale invariance of the process.

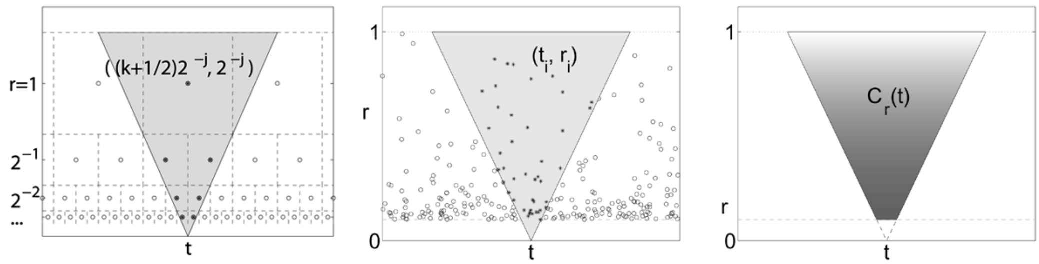

Fig. 1. Comparison between the "time-scale" construction of multiplicative cascades. Left: Nested geometry behind the binomial cascade, Center: Stationary discrete geometry behind the CPC, Right: Stationary continuous geometry behind the IDC. The shaded cones indicate the regions that determine the value of the cascade at time t.

the MPCP (or CPC) cascade reads as

$$\tilde{Q}_r(t) = \prod_{(t_i,r_i)\in\mathcal{C}_r(t)} W_i, \qquad Q_r(t) = \frac{\tilde{Q}_r(t)}{\mathbb{E}[\tilde{Q}_r(t)]}. \tag{4}$$

Note that the binomial cascade uses similarly all multipliers $W_{j,k}$ such that $((k+1/2)2^{-j}, 2^{-j})$ belong to some cone above time t.

To obtain simple iterative scaling laws for the MPCP (or CPC) one ensures that each "exponential frequency band" of scales between 2^{-j-1} and 2^{-j} contributes on the average the same number of multipliers to $Q_r(t)$. This results in an expected number of Poisson points in $\mathcal{C}_r(t)$ proportional to $-\log r$, just as for binomial cascade. Power laws in the form of (1) with bounded $C_q(\cdot)$ are then recovered, together with the powerful multifractal formalism [18].

Noting that compound Poisson distributions are infinitely divisible, it seems only too natural to generalize the CPC to the IDC by generalizing (4) to the form

$$\tilde{Q}_r(t) := \exp M(\mathcal{C}_r(t)) \tag{5}$$

with a continuous infinitely divisible random measure $dM(t,r)$ (see Section II-B and Appendix I). This was done in previous works [21]–[23] which dealt with the scale-invariant case yielding power-law scaling.

In this paper, we are mainly interested in scalings that involve *departures from exact power laws*. As we observed in the CPC case, power laws go hand in hand with an average number of multipliers in the cone $\mathcal{C}_r$ proportional to $-\log r$. This suggests to abandon the idea of statistically identical contributions from exponential frequency bands in order to find non-power-law scaling; as we will see, this comes at the cost of simple iterative arguments.

B. IDC: Basic Notions

We recall now the definition of the *IDC noise* [21]–[23], [25] which generalizes the CPC (4) of Barral and Mandelbrot [18] as well as ideas of Schmitt and Marsan [17]. Note that IDC are closely related to the Lévy stable chaos of Fan [30]. Note also that IDC enter the framework of the T-martingales of Kahane [31].

To this end, let G be an infinitely divisible distribution (see Appendix I) with moment generating function

$$\tilde{G}(q) := \exp[-\rho(q)] := \int \exp[qx]dG(x). \tag{6}$$

Let $dm(t,r) = g(r)dtdr$ be a positive measure on the time-scale half-plane $\mathcal{P}^+ := \mathbb{R}\times\mathbb{R}^+$. Let M denote an infinitely divisible, independently scattered random measure distributed by G, endowing the time-scale half-plane $\mathcal{P}^+$ and associated to its so-called control measure $dm(t,r)$ (see (66) in Appendix II). In particular, we have for any Borel set $\mathcal{E}$

$$\begin{aligned}\mathbb{E}[\exp[qM(\mathcal{E})]] &= \exp[-\rho(q)m(\mathcal{E})]\\ &= \exp\left[-\rho(q)\int_{\mathcal{E}} dm(t,r)\right].\end{aligned} \tag{7}$$

The specific choice of a time-invariant *control measure* $dm(t,r) = g(r)dtdr$ is not essential to a valid definition but is added to ensure stationarity of Q_r.

Finally, a cone of influence $\mathcal{C}_r(t)$ is defined for every $t\in\mathbb{R}$ as (see Fig. 1)

$$\mathcal{C}_r(t) := \{(t',r') : r\le r'\le 1, t-r'/2\le t' < t+r'/2\}. \tag{8}$$

Choosing the large scale in the cone $\mathcal{C}_r(t)$ equal to 1 is arbitrary and amounts to a simple choice of time and scale units. Furthermore, the symmetry of the cone's shape inflicts a causal as well as an anticipating component. Scaling results presented below extend without restriction to a purely causal version such as

$$\mathcal{C}_r(t) = \{(t',r') : r\le r'\le 1, t-r'\le t'\le t\}.$$

Definition 1: An *IDC noise* is a family of processes $Q_r(t)$ parameterized by r of the form (see Fig. 3)

$$\begin{aligned}Q_r(t) &= \frac{\exp M(\mathcal{C}_r(t))}{\mathbb{E}[\exp M(\mathcal{C}_r(t))]}\\ &= \exp[\rho(1)m(\mathcal{C}_r(t))]\exp M(\mathcal{C}_r(t)).\end{aligned} \tag{9}$$

In the light of the CPC of the previous section, the IDC noise can be interpreted as a "continuously iterative" multiplication (compare Fig. 1 (left) and (right)). An immediate consequence of the definition is that Q_r is a positive stationary random process with

$$\mathbb{E}Q_r = 1. \tag{10}$$

The key property (7) reminds of (2) in how it separates the dependence of the moment order q; it lies at the origin of all scaling properties obtained in the sequel. Indeed, the distribution G controls the structure function through ρ, while the control measure m and the shape of the cone $\mathcal{C}_r(t)$ set the speed of the cascade. Choosing $m(\mathcal{C}_r)$ proportional to $-\log r$ will lead to power laws. Aiming at non-power-law behaviors, one may explore these degrees of freedom offered by the cone and the control measure. However, one should keep in mind that a change in the choice of $dm(t,r)$ can be expressed equivalently by an appropriate change in the shape of the cone $\mathcal{C}_r(t)$.

C. IDC Noise: First Properties

A direct consequence of the infinite divisibility of the random measure M and (9) is that $Q_r(t)$ has a log-infinitely divisible distribution, that is, $\log Q_r(t)$ has an infinitely divisible distribution. Also, the IDC noise adheres to a form of exact scaling, however, not in the sense of (1) but with respect to the index r. It is convenient to set

$$\varphi(q) := \rho(q) - q\rho(1) = -\log\left(\frac{\mathbb{E}[e^{qX}]}{\mathbb{E}[e^X]^q}\right) = -\log\left(\frac{\mathbb{E}[Z^q]}{(\mathbb{E}[Z])^q}\right) \quad (11)$$

whenever defined, where $Z = \exp[X]$ and where X is distributed according to the infinitely divisible law G. Note that $\varphi(1) = 0$ by definition and that $\varphi(q) \leq 0$ for all $q > 1$ for which it is defined. To see this, recall the well-known fact that characteristic functions are log-convex. Consequently, ρ is concave and so must be φ due to (11). Since $\varphi(0) = \varphi(1) = 0$ the claim follows.

Lemma 1: Let Q_r be an IDC noise. Then

$$\mathbb{E}\left[Q_r^q\right] = \exp\left[-\varphi(q)m(\mathcal{C}_r)\right] = \exp\left[-\varphi(q)\int_r^1 ug(u)du\right]. \quad (12)$$

The fact that the distribution G underlying the IDC noise in Definition 1 is infinitely divisible is key to ensure the separation of the dependence of the moments $\mathbb{E}\left[Q_r^q\right]$ on order q and resolution r (12). Power-law behaviors will be recovered for the *scale-invariant* case defined by [18], [21]–[23]

$$m(\mathcal{C}_r(t)) = \begin{cases} -c \cdot \log r, & \text{for } 0 \leq r \leq 1 \\ 0, & \text{for } 1 \leq r. \end{cases} \quad (13)$$

or equivalently

$$dm(t,r) = \begin{cases} \frac{c}{r^2} dr dt, & \text{for } 0 < r \leq 1, \\ 0, & \text{for } 1 \leq r. \end{cases} \quad (14)$$

Equation (12) becomes

$$\mathbb{E}[Q_r(t)^q] = r^{c\varphi(q)}. \quad (15)$$

Interestingly, the correlations of IDC noises—and in fact all finite-dimensional distributions [23]—are entirely determined by the geometry of the cascade in terms of the intersection of cones in the time scale plane (see Fig. 2):

Lemma 2:

$$\mathbb{E}[Q_r(t)Q_r(s)] = \exp\left[-\varphi(2)m(\mathcal{C}_r(s) \cap \mathcal{C}_r(t))\right]. \quad (16)$$

In the scale-invariant case of (13), (16) becomes

$$\mathbb{E}[Q_r(t)Q_r(s)] = \begin{cases} |t-s|^{c\varphi(2)} e^{-c\varphi(2)(|t-s|-1)}, & \text{for } r \leq |t-s| \leq 1 \\ 1, & \text{for } 1 \leq |t-s|. \end{cases} \quad (17)$$

Equation (17) approximately behaves as a power law for small values of $r < |t-s| \ll 1$, while the finite scale effects are rendered by the exponential term for $|t-s| \simeq 1$. Note that an exact power-law behavior for all $r < |t-s| \leq 1$ may be recovered in (17) by adding a Dirac distribution $c\delta(r-1)$ to $g(r) = c/r^2$ in the definition of $dm(t,r)$. Indeed, we have then

$$m(\mathcal{C}_r(s) \cap \mathcal{C}_r(t)) = -c\log|t-s|. \quad (18)$$

This corresponds to the choice made by Bacry and Muzy [21] to build *exactly* scale-invariant measures.

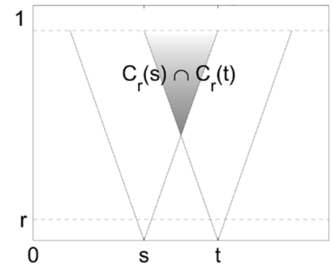

Fig. 2. The dependence between $Q_r(t)$ and $Q_r(s)$, in particular their correlation, stems entirely from the measure of the intersection of two cones $\mathcal{C}_r(t)$ and $\mathcal{C}_r(s)$.

D. IDC Noise: Examples

Infinitely divisible distributions G that may be used in this construction are often among well-known common distributions [32]. Most of these models have already been proposed for the modeling of intermittency in the context of turbulence in fluid mechanics [9]. Note that all expressions of $\varphi(q) = \rho(q) - q\rho(1)$ are constrained by the normalization of Q_r that imposes $\varphi(1) = 0$. Prominent examples include the following.

Example 1 (Normal): The underlying distribution G of the random measure M is normal, i.e., $\mathcal{N}(\mu, \sigma^2)$ [33]. Then, $\rho(q) = -\mu q - \frac{\sigma^2}{2}q^2$, so that $\varphi(q) = \rho(q) - q\rho(1)$ depends only on one parameter:

$$\varphi(q) = \frac{\sigma^2}{2}q(1-q). \quad (19)$$

Example 2 (Stable): The underlying distribution G of the random measure M is stable, i.e., $S(\alpha, \sigma, \mu, \beta)$ [34]. Such a choice would correspond to the model of turbulence proposed in [35]. Due to the heavy tails of G, normalizing the associate IDC noise (9) is only possible within a special range of parameter values. Indeed, choosing the tail exponent $\alpha < 1$ and the skewness parameter $\beta = -1$ then the random variable X with law G is almost surely smaller than the position parameter μ, and the Laplace transform $\mathbb{E}[\exp[qX]]$ is meaningful for $q > 0$ [34, Proposition 1.2.12]. One finds

$$\rho(q) = -\mu q - \sigma^\alpha |q|^\alpha \left(1 + \beta \operatorname{sign}(q) \tan\left(\frac{\pi\alpha}{2}\right)\right)$$

with $0 < \alpha < 2$, $(\alpha \neq 1)$, and $\beta = -1$, so that

$$\varphi(q) = \sigma^\alpha(q - q^\alpha)\left(1 - \tan\left(\frac{\pi\alpha}{2}\right)\right), \qquad \text{for } q > 0. \quad (20)$$

Example 3 (Gamma): The underlying distribution G of the random measure M is Gamma [32], [36]. Setting its parameters as $\alpha > 0$ and $\beta > 0$, we find $\rho(q) = \alpha\log(1 - q/\beta)$ which yields

$$\varphi(q) = \alpha\log\left(\frac{\beta - q}{\beta}\right) - \alpha q\log\left(\frac{\beta - 1}{\beta}\right). \quad (21)$$

Example 4 (Compound Poisson): CPC originally proposed in [18], [24] (see also [37]) were set as a special class of IDC noise in Section II-A. Since the multipliers $\{W_i\}_i$ are i.i.d. positive random variables independent of the point process $(t_i, r_i)_i$ (see (4)), the underlying distribution G of the random measure

$$dM(t,r) = \sum_{i:(t_i,r_i)\in dt\times dr} \log[W_i]$$

is here the compound Poisson distribution associated with the common distribution F of the $\log[W_i]$. The Laplace transform of F simplifies to $\tilde{F}(q) = \mathbb{E}[W^q]$; the compound distribution of F is given by $\tilde{G}(q) = \exp[\lambda(\tilde{F}(q) - 1)]$ where λ stands for the expected number of Poisson points. Since we can absorb λ into the control measure m, we may assume $\lambda = 1$ and set $\rho(q) = 1 - \mathbb{E}W^q$ and

$$\varphi(q) = 1 - \mathbb{E}[W^q] - q(1 - \mathbb{E}[W]). \tag{22}$$

Alternatively, an explicit computation of (12) confirms this form of φ up to a constant which can be absorbed into the control measure m.

Example 5 (Pure Poisson): The well-known She-Levêque model [38], [39] in the field of turbulence proposes a scaling with pure Poisson distributions. This can be realized with the simplest compound Poisson cascade where W reduces to constant W_0 with Dirac distribution F and an underlying pure Poisson distribution G of the random measure M. Noting $\mathbb{E}[W^q] = W_0^q$, (22) becomes

$$\varphi(q) = (1 - W_0^q) - q(1 - W_0). \tag{23}$$

E. Additional Comments

1) Self-Similar Processes: From the point of view of the analysis of scaling, self-similar processes can be seen as a particular case of multiscaling processes (cf. (1)). However, we emphasize that self-similar processes cannot be directly built as IDC noises. Self-similarity would correspond to a linear dependence of $\rho(q)$ on q of the form $\rho(q) = aq$. It corresponds to $G = \delta_a$ and $\varphi(q) = 0$ which describes a trivial cascade that yields a constant IDC noise $Q_r(t) = 1$. There is indeed no cascade of multipliers in this case.

2) On the Notions of Resolution r and Scale τ: We draw the attention of the reader to potential confusions regarding the true nature of the parameter r entering the definition of an IDC noise Q_r. Considering formal analogies between (2) and (12), for instance, r seems to play a role equivalent to τ and therefore can be considered as a *scale*. However, it should better be analyzed as a parameter of resolution: Q_r is the intermediate stage of a construction that evolves with parameter r whereas τ is the *scale* over which variations of $Q_r(t)$ may be analyzed.

III. Infinitely Divisible Cascading Motion

While Section II-A lays out the basis for continuous-scale or infinitely divisible multiplication, this section concerns the limiting behavior of the cascading noise $Q_r(t)$ as $r \to 0$ which makes it necessary to introduce its distribution function or primitive $A_r(t) = \int_0^t Q_r(s)ds$ as well as the scaling behavior of the limiting process A of A_r as $r \to 0$. Despite their appearing rather formal and mathematical, the questions of the behavior of Q_r and A_r as $r \to 0$ become crucial in practice. Indeed, numerical simulations become efficient and interesting in case of convergence only. Sections III-A and -B deal with these problems of convergence and define the infinitely divisible cascading motion. Then Section III-C states our principal results on the scaling behavior of A. These results are discussed in Section III-D and two non-scale-invariant examples are given in Section III-E.

A. Definition (Almost Sure Convergence)

The IDC noise inherits a powerful martingale property directly through the underlying infinitely divisible construction. Martingale techniques have traditionally been used to establish weak convergence of cascades ever since the celebrated work on T-martingales by Kahane and Peyrière [16], [28]. For convenience, set for $r < s < 1$

$$Q_r^s(t) := \exp\left[\rho(1)m(\mathcal{C}_r(t)\backslash\mathcal{C}_s(t))\right]\exp\left[M(\mathcal{C}_r(t)\backslash\mathcal{C}_s(t))\right] \tag{24}$$

and note

$$Q_r(t) = Q_r^s(t)\cdot Q_s(t) \tag{25}$$

where $Q_r^s(t)$ and $Q_s(t)$ are independent, and both of mean 1. Indeed, this is a simple consequence of (7), (9), and of M being independently scattered. Denoting by $\mathcal{F}_s$ the filtration induced by the process $Q_s(\cdot)$ it follows, still for $r < s < 1$, that

$$\mathbb{E}[Q_r(t)|\mathcal{F}_s] = \mathbb{E}\left[Q_r^s(t)\right]\cdot Q_s(t) = Q_s(t). \tag{26}$$

Thus, $\{Q_r(t)\}_{r>0}$ forms a continuously indexed martingale for each t. Furthermore, it is left-continuous, meaning that $Q_r(s) \to Q_s(t)$ as $r \uparrow s$ since $\cap_{r<s}\mathcal{C}_r(t) = \mathcal{C}_s(t)$ due to (8). Thus, $Q_{1/u}(t)$ is a right-continuous martingale, where we are interested in the limit $u \to \infty$; this corresponds to the traditional setting of the martingale convergence theorem which we recast here according to our setting.

Lemma 3: An IDC noise $\{Q_r(t)\}_{r>0}$ forms a positive, left-continuous martingale. Thus, it converges almost surely as $r \downarrow 0$.

Invoking the Law of Large Numbers it is then easy to show that $Q_r(t)$ converges almost surely and for almost all t to zero in many cases of interest such as the scale-invariant cascades. Indeed, $\mathbb{E}[\log Q_r(t)]$ is strictly negative due to Jensen's inequality, except in trivial cases. In rare places t, the noise will diverge to infinity (see Fig. 3) keeping on average a reasonable total mass when interpreted as a density. Motivated by this degeneracy of the limit of Q_r (see Lemma 3) and by analogy with the binomial

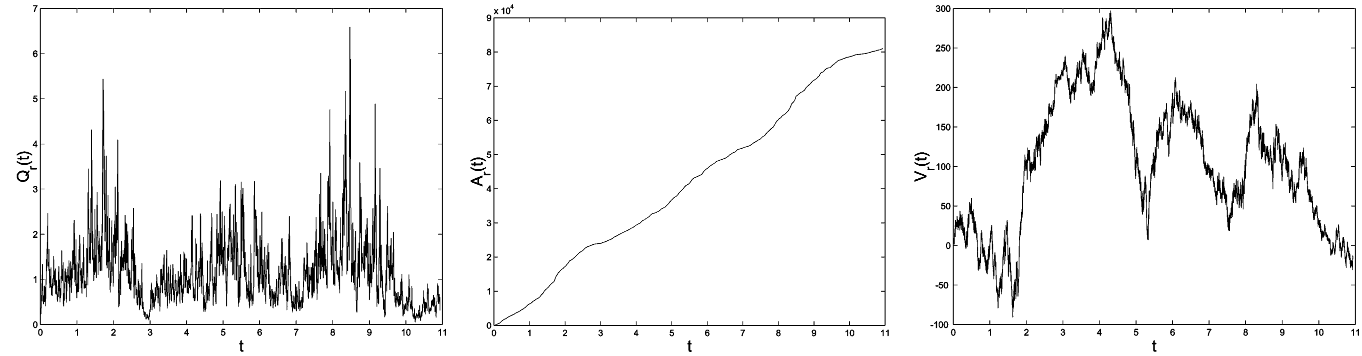

Fig. 3. Sample of a realization of $Q_r(t)$ (left), $A(t)$ (middle), and $V_H(t)$ (right).

cascades [14] and the theory of T-Martingales [28], we introduce the *infinitely divisible cascading motion* from the distribution function of the noise

$$A_r(t) = \int_0^t Q_r(s)ds. \tag{27}$$

Note that

$$\mathbb{E}[A_r(t)] = \int_0^t \mathbb{E}[Q_r(s)]ds = t. \tag{28}$$

The following lemma permits to properly define the limiting process obtained in the limit $r \to 0$.

Lemma 4: Let $Q_r(t)$ denote an IDC noise. There exists a cadlag (continuous from the right, limits from the left) process $A(\cdot)$ with stationary increments such that almost surely

$$A(t) = \lim_{r \to 0} A_r(t) \tag{29}$$

for all rational t simultaneously. This process A is called *IDC motion*.

Proof: Since conditional expectations commute with integrals, $\{A_r(t)\}_r$ forms for every t a positive, left-continuous martingale with respect to the filtration induced by $Q_r(\cdot)$. It converges, thus, almost surely for all rational t simultaneously. Since $Q_r > 0$, all A_r and A are nondecreasing and have limits from the left and right; thus, A can be extended to all real t by making it continuous from the left. □

The increment process $\delta_\tau A_r(t)$ of A_r

$$\delta_\tau A_r(t) = A_r(t+\tau) - A_r(t) = \int_t^{t+\tau} Q_r(s)ds \tag{30}$$

inherits full stationarity from Q_r. Recall that stationarity of Q essentially roots in the time invariance of both the control measure m and the shape of the cone $\mathcal{C}_r$.

B. Convergence in $\mathcal{L}^q$

While almost sure convergence is convenient to ensure a general definition, one requires the existence of moments to study scaling behavior. In addition, nothing assures *a priori* that A in (29) does not degenerate to zero itself. However, convergence in $\mathcal{L}^q$ for some $q > 1$ allows to conclude $\mathbb{E}[A(t)] = t$ from (28) and implies the nondegeneracy of A. To our knowledge, there is no general result available for $q > 2$ and only $1 < q \leq 2$ will be considered here. In the scale-invariant case, finer results for finiteness of moments of positive oreders can be found in [18], [21], [23].

The simplest such convergence criterion is in terms of a second-order analysis and follows from standard facts on $\mathcal{L}^2$-bounded martingales.

Proposition 1: An IDC motion A_r converges in $\mathcal{L}^2$ if and only if there exists some finite constant K such that for all $r > 0$

$$\mathbb{E}A_r(t)^2 = \int_0^t \int_0^t \exp\{-\varphi(2)m(\mathcal{C}_r(u) \cap \mathcal{C}_r(v))\}\, dudv < K. \tag{31}$$

When this condition is verified, $A(t)$ is nondegenerate and $\mathbb{E}[A(t)] = t$.

In the scale-invariant case (13), explicit computation renders the criterion (31) equivalent to $c\varphi(2) + 1 > 0$ (see [18], [23]). Recall that $\varphi(2)$ is always negative—see (11). It is an easy exercise to verify this claim by explicit computation for the CPC with multipliers W of mean one where $-\varphi(2)$ is simply the variance of the multipliers.

Going into more mathematical details, a more general criterion is obtained by extending a theorem by Barral [26, Theorem 6] as follows.

Proposition 2: Let $1 < q \leq 2$. Fix $t > 0$. Assume that there exists an integer $k_o \geq 2$ such that

$$\sum_{n\geq 0} k_o^{-n(1-1/q)} \left(\mathbb{E}\left[Q^q_{tk_o^{-n-1}}(t)\right]\right)^{1/q} = \sum_{n\geq 0} k_o^{-n(1-1/q)} \exp[-m(\mathcal{C}_{tk_o^{-n-1}}) \cdot \varphi(q)/q] < \infty. \tag{32}$$

Then, $A_r(t)$ converges almost surely and in $\mathcal{L}^q$.

Proof: First, one needs to extend, in fact, Lemma 3 of [26] from CPC to arbitrary IDC. This is done by using the auxiliary IDC cascade induced by $dM'(t,r) = p \cdot dM(t,r)$ which is obtained by rescaling the underlying measure M of the original IDC by the constant p. For a CPC, this amounts to replacing the positive multipliers W by W^q as done in [26]. Second, one verifies that the assumptions of [26, Theorem 6(ii)] hold by exploiting the time-invariance of the cones used here. □

Corollary 1: Let $1 < q \leq 2$. A sufficient condition for convergence of $A_r(t)$ in $\mathcal{L}^q$ is

$$\limsup_{n\to\infty} \frac{1}{n} m(\mathcal{C}_{tk_o^{-n-1}}) < \frac{q-1}{\varphi(q)} \log(1/k_o) \tag{33}$$

for some integer $k_o \geq 2$ (recall that $\varphi(q) < 0$). In the scale-invariant case of $m(\mathcal{C}_r) = -c\log(r)$ this becomes

$$(q-1) + c\varphi(q) > 0. \quad (34)$$

Such criteria have been obtained in [23] for scale-invariant IDC, and in [26] for CPC. As will follow from the scaling properties of A, they are quasi-tight for certain IDC (see Corollary 3).

C. Scaling Properties of an IDC Motion

This subsection outlines our main theoretical results which characterize the scaling properties of non-scale-invariant IDC motions. Note that currently existing results cover the scale-invariant case only [21]–[23]. For the sake of uninterrupted reading, the proofs have been postponed to Appendix III. Only some key points are given in this subsection.

Our approach exploits the rescaling property of IDCs, inspired by the scale-invariant case for which $Q_r^{b^n}(t)$ is equal in distribution[3] to $Q_{r/b^n}(t/b^n)$, where $b < 1$ and $r < b^n$. We start by making this precise. The recursion (25) between the Q_r^b translates into a recursion between distribution functions: set $A_{r/b}^{(1)}(t/b) := (1/b)\int_0^t Q_r^b(s)ds$. Simple plug and play yields the following.

Lemma 5: Let $0 < r \leq b < 1$. Then $A_r^{(1)}$ is independent of Q_b and

$$A_r(t) = \int_0^t Q_b(s)Q_r^b(s)ds \quad (35)$$

$$= b\int_0^t Q_b(s)d\left[A_{r/b}^{(1)}\left(\frac{s}{b}\right)\right]. \quad (36)$$

Iterating this idea we set for $r < b^n$

$$A_{r/b^n}^{(n)}(t) := \frac{1}{b^n}\int_0^{b^n t} Q_r^{b^n}(s)ds = \int_0^t Q_r^{b^n}(b^n s)ds = \int_0^t Q_{r/b^n}^{(n)}(s)ds. \quad (37)$$

Clearly, $A^{(n)}$ is again a cascading motion. Let $m^{(n)}$ denote the control measure associated to $Q_{r/b^n}^{(n)}(s)$. Then, by (37)

$$m^{(n)}(\mathcal{C}_{r/b^n}(t)) = m\left(\mathcal{C}_r^{b^n}(b^n t)\right)$$

where in analogy to (25) we set

$$\mathcal{C}_r^{b^n}(s) = \mathcal{C}_r(s)\backslash\mathcal{C}_{b^n}(s).$$

As a consequence, $dm^{(n)}(t,r) = g^{(n)}(r)dtdr$ with

$$g^{(n)}(r) := b^{2n}g(b^n r)\cdot \mathbf{1}_{[0,1]}. \quad (38)$$

Indeed, simple substitution yields

$$m\left(\mathcal{C}_r^{b^n}(0)\right) = \int_r^{b^n} ag(a)da = \int_{r/b^n}^1 b^n a' b^n g\left(b^n a'\right)da' = m^{(n)}(\mathcal{C}_{r/b^n}(0))$$

[3]Notably, this property in distribution is lost in the *exact* power-law scaling case of (18) studied in [21], [23] where a different approach is used.

which confirms (38). We may understand $A_r^{(n)}$ as a zoom into the small scale details of the construction of A_r. Indeed, in the scale-invariant case ($g(r) = c/r^2$), we have $g^{(n)} = g$, thus,

$$Q_r^{b^n}(b^n\cdot) \stackrel{\text{fdd}}{=} Q_{r/b^n}(\cdot)$$

and

$$A_r^{(n)}(t) = \int_0^t Q_{rb^n}^{b^n}(b^n s)ds \stackrel{\text{fdd}}{=} A_r(t).$$

If the integrand Q_b in (35) were constant over the interval $[0,t]$ we could pull it out of the integral and a scaling law of moments would immediately follow. A measure for the variation of the integrand which will prove useful is the following (see Appendix III):

$$\Delta_{b,q}^{(n)}(t) := \frac{\mathbb{E}\sup_{0\leq s\leq b^n t}\left|Q_{b^n}^{b^{n-1}}(s)^q - Q_{b^n}^{b^{n-1}}(0)^q\right|}{\mathbb{E}\left[Q_{b^n}^{b^{n-1}}(0)^q\right]}. \quad (39)$$

Thus, our main result which is established in Appendix III reads as follows (see also Appendix IV for a direct derivation for $q = 2$ in a specific case).

Theorem 1:
Fix $q > 0$, $b \in (0,1)$, $\rho(\cdot)$ and $dm = g(r)dtdr$.

(Moment condition)	Assume that A_r converges in $\mathcal{L}^q$.
(Variational condition)	Assume there exists $\nu > 0$ such that $\Delta_{b,q}^{(n)}(t) \leq C_{b,q}t^\nu$ for all $n \in \mathbb{N}$ and all $0 < t \leq 1$.
(Speed condition)	Assume that $g^{(n)}$ converges. Then there exist constants $\overline{C}_q$ and $\underline{C}_q$ such that for any $t < 1$

$$\underline{C}_q t^q \exp\left[-\varphi(q)m(\mathcal{C}_t)\right] \leq \mathbb{E}A(t)^q \leq \overline{C}_q t^q \exp\left[-\varphi(q)m(\mathcal{C}_t)\right]. \quad (40)$$

We emphasize that such a scaling behavior permits for the first time to observe controlled departures from the standard power-law behavior over a continuous range of scales. Playing with the form of $m(\mathcal{C}_t)$, one may obtain a variety of situations. This is illustrated in Sections III-E and V. Moreover, the stationarity of increments has been maintained. Note that such a non-scale-invariant approach implies some specific technical difficulties. Clearly, the assumptions simplify drastically in the scale-invariant case since $g^{(n)} = g$ for all n, since Δ does not depend on n, and since (34) holds.

The *speed condition* could be relaxed to require that the $g^{(n)}$ are bounded; however, this would entail technical subtleties in the proofs.

The *variational condition* displays a rather technical aspect but is actually satisfied for any normal and certain compound poisson cascades according to the following corollary.

Corollary 2: Assume that A_r is either a normal IDC motion or a CPC with $\mathbb{E}[W^q] < \infty$. Assume that $g^{(n)}$ converges. Then, the variational condition $\Delta_{b,q}^{(n)}(t) \leq C_{b,q}t^\nu$ of Theorem 1 holds.

As a consequence, Theorem 1 can be applied to a wide variety of IDCs, including normal cascades as well as CPCs with $\mathbb{E}[W^q] < \infty$.

The *moment condition* was dealt with in Proposition 2 and Corollary 1 (for $1 < q \leq 2$). As a particular consequence of the scaling law, we find that the sufficient condition for convergence in $\mathcal{L}^q$ of Corollary 1 is quasi-tight.

Corollary 3: Let $1 < q \leq 2$. Assume that (40) holds. A necessary condition for convergence of $A_r(t)$ in $\mathcal{L}^q$ is

$$\limsup_{n\to\infty} \frac{1}{n} m(\mathcal{C}_{tk_o^{-n-1}}) \leq \frac{q-1}{\varphi(q)} \log(1/k_o) \tag{41}$$

for all integers $k_o \geq 2$ (recall that $\varphi(q) < 0$).

In the scale-invariant case of $m(\mathcal{C}_r) = -c\log(r)$, this necessary condition becomes

$$(q-1) + c\varphi(q) \geq 0 \tag{42}$$

and was observed in [23]. The proof of [23, Lemma 3] generalizes easily to establish Corollary 3.

D. Discussion of Scaling Laws

We add a few remarks useful to applications (see also [27] for more details). Let us start by pointing out that in the case where $g^{(n)}$ actually converges, then its limit is necessarily the fixed point of the transformation $g(r) \to b^2 g(b^2 r)$ which is nothing but the scale-invariant case. Thus, for such cases, the cascade will show only some subdominant yet visible corrections to power laws (see examples in Section III-E). Since only boundedness of the tail of $g^{(n)}$ is required, some further flexibility is present.

Scaling of Increments: The fact that $A(t)$ has stationary increments and $A(0) = 0$ yields the useful scaling laws on the increments $\delta_\tau A$ of A

$$\mathbb{E}[\delta_\tau A^q] \sim C_q \tau^q \exp\left[-\varphi(q) m(\mathcal{C}_\tau)\right], \qquad \forall\, \tau \leq 1 \tag{43}$$

where "$\sim$" is used as a short notation for inequalities like in (40); in practice, it turns out that both sides of the "$\sim$" are close to proportional for $\tau \ll 1$. Moreover, one expects that $\mathbb{E}\left[\delta_\tau A^q\right] \sim \tau^q$ for large $\tau \gg 1$. This is in essence a consequence of the Law of Large Numbers: while for small τ, the noise is quite correlated, it decorrelates quickly as $\tau \gg 1$.

Now remember that, inspired by previous works [3], [5], we were *a priori* searching for non-power-law scaling of the form $\exp[-\zeta(q) n(\tau)]$ as in (2). Rather, through our approach we are naturally led to a mixture of a power-law and a non-power-law behavior of the form $\tau^q \cdot \exp[-\varphi(q) m(\mathcal{C}_\tau)]$. This result is inherent to the use of an integral to define $A(t)$. On one hand, the $\exp\left[-\varphi(q) m(\mathcal{C}_\tau)\right]$ term is related to the underlying IDC noise. On the other hand, the τ^q term is due to the fact that an IDC motion is obtained by *integration* of an IDC noise.

Locally Averaged IDC Noise: In many applications, Q_r would be the quantity of interest for modeling: dissipation in turbulence, packet flows in Internet traffic, numbers of transactions in finance, etc.Then a classical analysis consists in studying box averages over varying time lags. Thus, such an analysis focuses on $\delta_\tau A/\tau$. In view of (40) obtained in Theorem 1 or, equivalently, in view of (43), we are led to consider the following process:

$$\frac{1}{\tau}\int_t^{t+\tau} Q_r(s)ds = \frac{1}{\tau}\delta_\tau A_r(t) = \frac{1}{\tau}(A_r(t+\tau) - A_r(t)) \tag{44}$$

that can be read either as a locally averaged IDC noise or as normalized increments of the IDC motion. From earlier sections, one has that $\frac{1}{\tau}\int_t^{t+\tau} Q_r(s)$ scales like

$$\mathbb{E}\left(\frac{1}{\tau}\int_t^{t+\tau} Q_r(s)ds\right)^q \sim \exp[-\varphi(q) m(\mathcal{C}_\tau)]. \tag{45}$$

Thus, infinitely divisible cascades provide us with a versatile family of models that allow for a variety of scale dependence. Such behavior is to be compared to (2) which shows that the process $\frac{1}{\tau}\int_t^{t+\tau} Q_r(s)$ meets the requirement of separation of the form $\exp[-\zeta(q) n(\tau)]$ between variables τ and q. Comparing to (12) as well, we emphasize again a fundamental difference between r and τ. In (12), the dependence is on r while in (45), the dependence is on the scale variable τ. This latter case betrays a scaling phenomenon while the former does not. This difference is sometimes evoked in turbulence [35] by making the distinction between the *bare* cascade (Q_r here) and the *dressed* cascade ($\delta_\tau A_r/\tau$ here).

Back to the original ideas of Mandelbrot [14], when he introduced conservative cascades for the modeling of dissipation in turbulence, one can read Q_r as the dissipation function as measured at Kolmogorov length η (η then corresponds to r), while $\frac{1}{\tau}\int_t^{t+\tau} Q_r(s)$ stands for the aggregated dissipation in *boxes* of *scale* τ. The scaling behavior of this latter quantity has been widely studied in experimental hydrodynamics turbulence (see, e.g., [40] for a review).

Continuous Multiscaling: A key property of these scaling behaviors (40) or (43) is that they hold continuously through the scales, not only for a particular set of discrete scales. Again, we put the emphasis as well on the fact that the construction of Q_r and A enables full control of the way the cascading process develops along scales and not only of the multifractal behavior obtained in the limit $\tau \to 0$. As far as applications and real-world data modeling are concerned, we believe that control of the entire cascade process is probably more relevant than that of the asymptotic behavior as $\tau \to 0$ only.

Numerical Simulations: In numerical simulations (see Section V), one has to deal with A_r rather than with the limiting process A since the limit $r \to 0$ remains out of reach. However, for sufficiently small r one has

$$\mathbb{E}\left[\delta_\tau A_r^q\right] \sim C_q \tau^q \exp\left[-\varphi(q) m(\mathcal{C}_\tau)\right], \qquad \forall\, r \leq \tau \leq 1. \tag{46}$$

Equation (46) clearly underlines the different status of time-lag τ and resolution r: r acts as a limiting *resolution* below which scaling properties are not controlled while τ stands for the *scale* at which the process is analyzed.

Scale-Invariant Case: Consistently, the power-law behaviors of the known scale-invariant case [21]–[23] are recovered as a corollary of Theorem 1.

Corollary 4: Let A_r be an IDC motion with scale-invariant control measure (13). Assume that A_r converges in $\mathcal{L}^q$ and that $\Delta_{b,q}^{(0)}(t) = O(t^\nu)$ as $t \to 0$ for some $\nu > 0$. Then

$$\underline{C}_q t^{q+c\varphi(q)} \leq \mathbb{E}A(t)^q \leq \overline{C}_q t^{q+c\varphi(q)}, \qquad \text{for } t < 1. \tag{47}$$

A further remarkable consequence follows from Kolmogorov's criterion.[4]

Corollary 5: Let A_r be an IDC motion with scale-invariant control measure. Then there exists a continuous version of A such that almost all paths have global Hölder regularity h for all $h < (q-1+c\varphi(q))/q$ for all values of q for which (47) holds.

E. Non-Scale-Invariant Examples

Since departures from power laws are one of the major goal of the present work, we give here two precise examples of non-scale-invariant measure $m(\mathcal{C}_r)$ for which we may verify the assumption of Theorem 1 and Corollary 2.

Example 6: Let us consider the following slight deviation from the scale-invariant case:

$$dm(t,r) = g(r)dtdr = \frac{c}{r^2\,(1+1/\log(\delta/r))}dtdr. \qquad (48)$$

Note first that the densities $g^{(n)}$ converge to the scale-invariant density c/r^2. Second, note that applying (33) leads to the same $\mathcal{L}^q$-convergence criterium as in the scale-invariant case, as it should

$$(q-1)+c\varphi(q) > 0. \qquad (49)$$

Moreover, it is a sufficient condition for $\mathcal{L}^q$-convergence for all cascades $A_r^{(n)}$. Despite the close approximation by the scale-invariant cascade, this example spots non-power-law progression of moments since

$$m(\mathcal{C}_\tau) = -c\log\tau + c\log\left(\frac{1+\log(\delta)}{1+\log(\delta/\tau)}\right). \qquad (50)$$

Example 7: This example is inspired from consideration in the analysis of hydrodynamic turbulence [5] where a dependence in $1-r^{-\beta}$ in place of $\log r$ was proposed to take into account the departures from power-law behaviors observed on empirical data. This choice results naturally as it provides a family of functions indexed by only one parameter β which tends to the function $\log r$ as $\beta \to 0$. Moreover, a direct computation of the behavior of $\mathbb{E}A(t)^2$ is possible and is reported in Appendix IV.

Modifying the scale-invariant case, we consider the measures $dm(t,r) = c/r^{2+\beta}dtdr$ to achieve the proposed scaling. The case $\beta > 0$ gives rise to a divergence as $r \to 0$ so that A_r does not converge to a meaningful process $A(t)$. Indeed, (33) yields nothing since the left-hand side is infinite.

When $\beta < 0$, $g^{(n)}$ vanishes identically in the limit. This is related to the fact that $\lim_{r\to 0} m(\mathcal{C}_r)$ is finite. As a consequence, the limit of $A_r \ (r \to 0)$ poses no problem and its multifractal behavior (in the limit $\tau \to 0$) is trivial. Let us add that $g^{(n)}$ converges to zero when $\beta < 0$ which simplifies the assumptions of Theorem 1 and Corollary 2. However, the non-power-law behavior at scale τ is controlled by

$$m(\mathcal{C}_\tau) = c\frac{1-\tau^{-\beta}}{-\beta} \qquad (51)$$

and remains interesting in a wide range of scales $\tau < 1$. This example is of particular interest and will be extensively used in Section V devoted to illustrations (see also Appendix IV).

The correction terms to the power law found in these examples may be subtle, yet they reflect true scaling and cannot be subsumed by a constant error bound (see Section V). To our knowledge, these are the first cascades which deviate from pure power-law scaling.

[4]Kolmogorov's criterion (see, for example, [41]) : If $\{X(t) : t \in R\}$ is a stochastic process with values in a complete separable metric space (S,d), and if there exists positive constants β, C, ϵ such that for all $s,t \in R$ we have

$$\mathbb{E}d(X_s,X_t)^\beta \leq C|s-t|^{1+\epsilon}$$

then there exists a continuous version of X. This version is Hölder continuous of order θ for each $\theta < \epsilon/\beta$.

IV. Infinitely Divisible Cascading Random Walk

By construction, A is a nondecreasing process and this can be seen as a severe limitation for the modeling of real-world data. As was already proposed in the scale-invariant case [21]–[23], following an idea which goes back to Mandelbrot [42] and to the Brownian motion in multifractal time, one can define a process with stationary increments, continuous scale invariance, prescribed departures from power laws, and prescribed scaling exponents as well as positive and negative fluctuations: the *infinitely divisible cascading random walk*, V_H.

A. Definition

Definition 3: Let A be an IDC motion, and B_H the fractional Brownian motion with Hurst parameter H, B_H being independent of A. The process (see Fig. 3)

$$V_H(t) = B_H(A(t)), \qquad t \in \mathbb{R}^+ \qquad (52)$$

is called an *IDC random walk*.

For practical use in simulations, we define

$$V_{H,r}(t) = B_H(A_r(t)). \qquad (53)$$

B. Scaling Properties

Using the self-similarity of B_H and the independence between B_H and A, one finds that

$$\mathbb{E}[|V_H(t)|^q] = \mathbb{E}\mathbb{E}[|B_H(A(t))|^q|A] \qquad (54)$$

$$= \mathbb{E}[|B(1)|^q]\cdot\mathbb{E}[|A(t)|^{qH}]. \qquad (55)$$

As an immediate consequence we get the following.

Theorem 2: Under the assumptions of Theorem 1, there exist constants $\overline{C}_q$ and $\underline{C}_q$ such that for any $t < 1$

$$\underline{C}_q t^{qH}\exp\left[-\varphi(qH)m(\mathcal{C}_t)\right] \leq \mathbb{E}[|V_H(t)|^q] \leq \overline{C}_q t^{qH}\exp\left[-\varphi(qH)m(\mathcal{C}_t)\right]. \qquad (56)$$

Theorem 2 calls for comments related to those concerning Theorem 1 (Section III-C). In particular, since both B_H and A have stationary increments, so does V_H. As a consequence, for $\tau < 1$, increments $\delta_\tau V_H$ of V_H obey[5]

$$\mathbb{E}\left[\delta_\tau V_H^q\right] \sim C_q\tau^{qH}\exp\left[-\varphi(qH)m(\mathcal{C}_\tau)\right]. \qquad (57)$$

For $\tau > 1$, it reduces to $\mathbb{E}\left[\delta_\tau V_H^q\right] \sim C_q\tau^{qH}$.

Thus, IDC random walks are processes with stationary increments that display non-power-law multiscaling prescribed

[5]Again, "$\sim$" is used as a short notation for inequalities like in (40) and (56).

a priori over a continuous range of scales as well as positive and negative fluctuations, see Fig. 3.

In the scale-invariant case for which results were already obtained in [21]–[23], (56) consistently reduces to

$$\underline{C}_q t^{qH+c\varphi(qH)} \le \mathbb{E}|V_H(t)|^q \le \overline{C}_q t^{qH+c\varphi(qH)}. \tag{58}$$

C. The Case of the Brownian Motion ($H = 1/2$)

This subsection focuses on the simplest case, namely, Brownian motion with $H = 1/2$, and introduces a process meant to mimic $V_{1/2}$. The process Z is defined by the limiting stochastic integral

$$Z(t) = \lim_{r\to 0} Z_r(t) \tag{59}$$

where

$$Z_r(t) = \int_0^t \sqrt{Q_r(s)}dB(s) \tag{60}$$

whenever it exists, with $\sqrt{Q_r(s)}dB(s)$ the corresponding IDC Gaussian noise; $Q_r(s)$ and $B(s)$ are independent. In contrast, with $A_r(t)$ obtained from a deterministic integral, the process $Z_r(t)$ appears as a stochastic integral of $Q_r(t)$. The process $Z_r(t)$ is indeed a *random walk*. In the scale-invariant case, it corresponds to the *MRW* introduced in [19].

Clearly, $Z_r(t)$ and $V_{1/2,r}(t)$ do not have equal paths. Consider all paths for which $Q_r(s)$ takes a constant value over a small interval, say over $[0,\varepsilon]$; note that this happens with positive probability in our framework. Setting $C = Q_r(0)$ for short, we find for $t \in [0,\varepsilon]$ that $V_{1/2,r}(t) = B(Ct)$ while $Z_r(t) = \sqrt{C}B(t)$. While these parts of the paths are obviously different, they are in fact still equal in finite-dimensional distributions (conditioned on the constant $Q_r(s)$).[6]

We establish the following proposition in Appendix V.

Proposition 3: For any $r \in [0,1]$, processes $V_{1/2,r}(t) = B(A_r(t))$ and $Z_r(t)$ are identical in the sense of finite-dimensional distribution. Furthermore

$$\mathbb{E}Z_r(s)Z_r(t) = \mathbb{E}V_{1/2,r}(s)V_{1/2,r}(t) = \sigma^2 \min(s,t).$$

Note that a definition of a process $Z_{H,r}$ generalizing (60) to the case $H \neq 1/2$ brings up the problem of a relevant definition of the stochastic integration with respect to the fractional Brownian motion which, to our knowledge, has not yet been properly solved in the general case.

V. Illustration

This section presents results obtained from numerical simulations of infinitely divisible cascading processes, respectively, in the exact scale-invariant case and in a non-scale-invariant case. We know from Theorems 1 and 2 that the exact scale-invariant case yields power-law behaviors of the moments of the increments of $A(t)$ and $V_H(t)$ while departures from power laws are expected in the non-scale-invariant case. This is illustrated below for CPCs as well as for log-normal cascades. Algorithms used to produce the samples shown here are described in detail in Section VI.

A. Parameters of Numerical Simulations

The following two sets of infinitely divisible cascading processes possess the same general characteristics apart from their control measure $dm(t,r)$: $dm(t,r) = c\,dt\,dr/r^2$ in the scale-invariant case, $c=3$; $dm(t,r) = c\,dt\,dr/r^{2+\beta}$ with $\beta = -0.4$ in the non-scale-invariant case, $c = 20$ (Example 2 of Section III-E). These choices[7] lead, respectively, to $m(\mathcal{C}_\tau) = -c\log\tau$ and $m(\mathcal{C}_\tau) = c(1-\tau^{-\beta})/(-\beta)$ for $\tau \le 1$. Only scales $\tau \le 1$ are influenced by $m(\mathcal{C}_\tau)$. Note that the scale-invariant situation is recovered from the non-scale-invariant one by taking the limit $\beta \to 0$.

Infinitely divisible cascading processes presented below are CPCs (see (4) and Section II-D). Distribution F of $\log W_i$ is a log-normal distribution with moment generating function $\tilde{F}(q) = \mathbb{E}[W_i^q] = \exp(\mu q + \sigma^2 q^2/2)$ so that

$$\varphi(q) = c(1-\exp(\mu q+\sigma^2 q^2/2)) - cq(1-\exp(\mu+\sigma^2/2)) \tag{61}$$

where $(\mu;\sigma^2) = (-0.1; 0.05)$. The Hurst exponent H of the fractional Brownian motion used to build $V_H(t)$ has been set to $H = 1/3$.

Many realizations are necessary to ensure statistical convergence of the (rudimentary) analysis procedure carried out here: we used about 1000 realizations of 2^{15} points corresponding to a total amount of about 3.10^7 points. In both cases, a τ^q (resp., τ^{qH}) term always dominates the behaviors of $\mathbb{E}[\delta_\tau A^q]$, see (40) (resp., $\mathbb{E}[\delta_\tau V^q]$, see (56)). As a consequence, the performed analysis focuses on the scaling behaviors of $\mathbb{E}[(\delta_\tau A/\tau)^q] \sim \exp[-\varphi(q)m(\mathcal{C}_\tau)]$, respectively, $\mathbb{E}[(\delta_\tau V/\tau^H)^q] \sim \exp[-\varphi(qH)m(\mathcal{C}_\tau)]$.

Remark: Fig. 4 shows the results obtained for CPCs. Fig. 5 shows similar results for log-normal cascades with $\sigma^2 = 0.2$ and the same choice for $dm(t,r)$ as above.

B. Scale-Invariant Cascade

The well-known scale-invariant case serves as a reference to emphasize what is obtained in the non-scale-invariant case. As expected, the moments of the increments of $A(t)$ and $V_H(t)$ obey power laws in a large range of scales $\tau < 1$. Indeed, Fig. 4(a) shows that $\mathbb{E}[(\delta_\tau A/\tau)^q]$ behaves like

$$\exp[-\varphi(q)m(\mathcal{C}_\tau)] = \tau^{c\varphi(q)}.$$

Exponents $\varphi(q)$ (resp., $\varphi(qH)$) estimated by linear regressions in log-log diagrams are consistent with expected theoretical values, see Fig. 6 (left). Similarly, Fig. 4(b) shows that $\mathbb{E}[(\delta_\tau V/\tau^H)^q]$ behaves like $\tau^{c\varphi(qH)}$. Estimated exponents are consistent with expected theoretical values, see Fig. 6 (right).

[6]Since Ito integrals are defined via $\mathcal{L}^2$ approximation of the integrand by step functions, this argument bears validity beyond our piecewise-constant cascades Q.

[7]The constant c has been set to $c = 3$ in the scale-invariant case and $c = 20$ in the non-scale-invariant case in order to respect the $\mathcal{L}^2$-convergence criterion of Proposition 1.

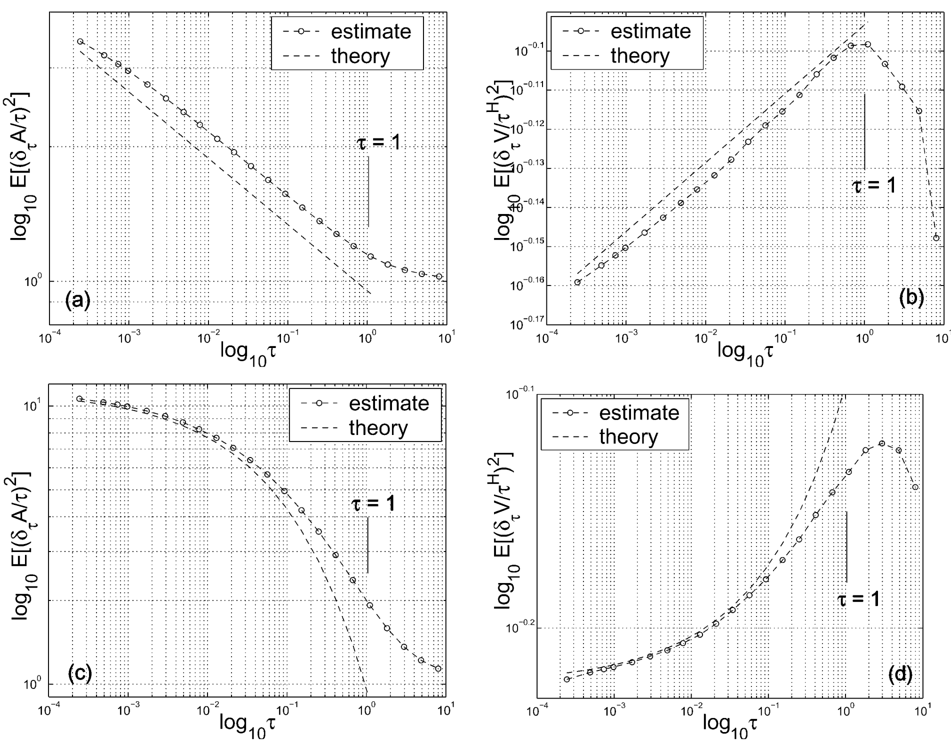

Fig. 4. CPCs: Scale-invariant cascades show power laws. (a) $\log \mathbb{E}[(\delta_\tau A/\tau)^2]$ compared to $c\varphi(2)\log\tau + Cte$. (b) $\log \mathbb{E}[(\delta_\tau V/\tau^H)^2]$ compared to $c\varphi(2H)\log\tau + Cte$. Non-scale-invariant cascade deviates from power laws. (c) $\log \mathbb{E}[(\delta_\tau A/\tau)^2]$ compared to $-\varphi(2)m(\mathcal{C}_\tau) + Cte$. (d) $\log \mathbb{E}[(\delta_\tau V/\tau^H)^2]$ compared to $-\varphi(2H)m(\mathcal{C}_\tau) + Cte$.

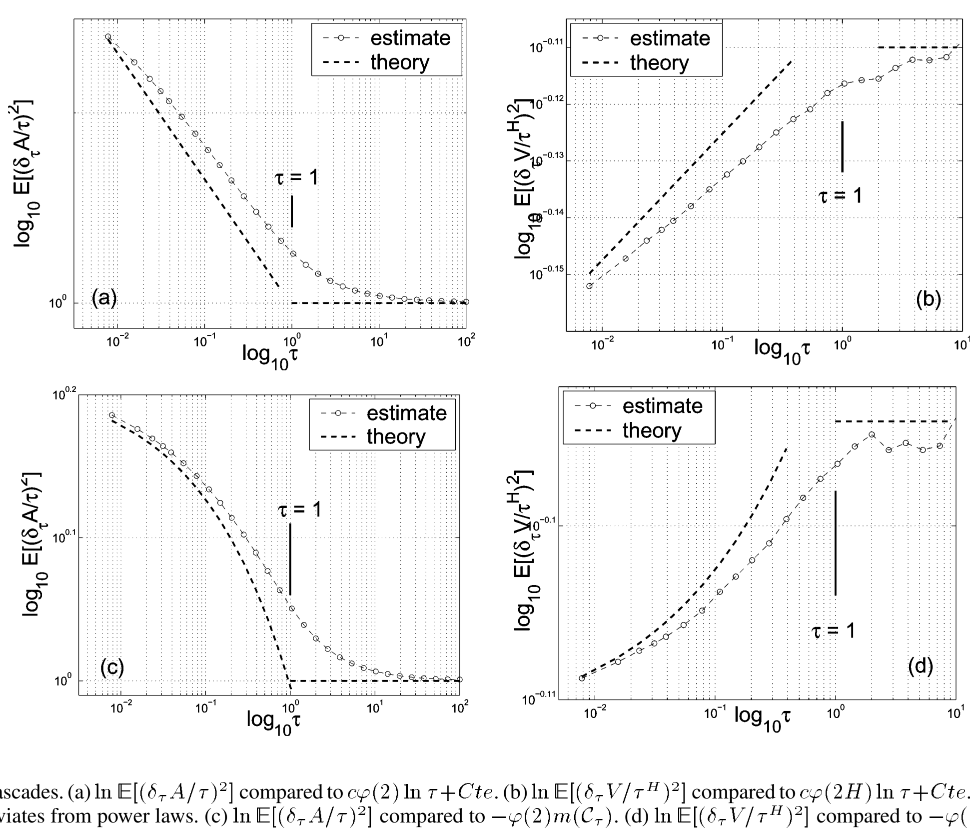

Fig. 5. Log-normal cascades. (a) $\ln \mathbb{E}[(\delta_\tau A/\tau)^2]$ compared to $c\varphi(2)\ln\tau + Cte$. (b) $\ln \mathbb{E}[(\delta_\tau V/\tau^H)^2]$ compared to $c\varphi(2H)\ln\tau + Cte$. The non-scale-invariant log-normal cascade deviates from power laws. (c) $\ln \mathbb{E}[(\delta_\tau A/\tau)^2]$ compared to $-\varphi(2)m(\mathcal{C}_\tau)$. (d) $\ln \mathbb{E}[(\delta_\tau V/\tau^H)^2]$ compared to $-\varphi(2H)m(\mathcal{C}_\tau)$.

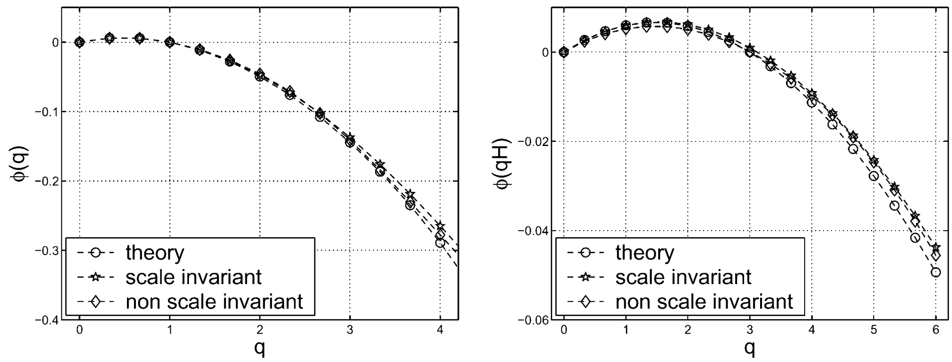

Fig. 6. CPCs. (Left) $\varphi(q)$ estimated from linear regressions in log-log diagrams of $\mathbb{E}[(\delta_\tau A/\tau)^q]$ versus $m(\mathcal{C}_\tau)$. (Right) $\varphi(qH)$ estimated from linear regressions in diagrams of $\mathbb{E}[(\delta_\tau V/\tau^H)^q]$ versus $m(\mathcal{C}_\tau)$.

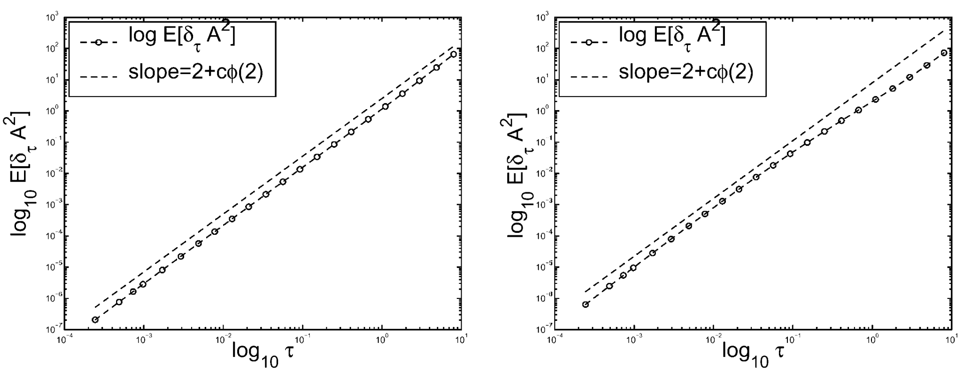

Fig. 7. (Left) Scale-invariant case obeys a very linear behavior that denotes an exact power-law scaling. (Right) Non-scale-invariant case slightly but significantly deviates from a power-law behavior: it is even directly visible in a $\log \mathbb{E}[(\delta_\tau A/\tau)^2]$ versus $\log \tau$ diagram. $\log \mathbb{E}[(\delta_\tau A/\tau)^2]$ versus $\log \tau$ does not exhibit a linear behavior even though it looks close to a linear behavior (compare (left) & (right)).

C. Non-Scale-Invariant Cascade

We now concentrate on the choice

$$m(\mathcal{C}_\tau) = c(1 - \tau^{-\beta})/(-\beta)$$

for $\beta = -0.4$. Therefore, departures from power-law behaviors corresponding to the $\exp[-\varphi(q)m(\mathcal{C}_\tau)]$ term in (40) are expected. Fig. 4(c) and (d) shows that such departures are observed on both $A(t)$ and $V_H(t)$. Compare to Fig. 4(a) and (b) corresponding to the scale-invariant case. It is remarkable that these departures are accurately controlled for $\tau < 1$ by the precise form of $m(\mathcal{C}_\tau) \neq -c\log\tau$. These numerical observations are perfectly consistent with the results of Theorems 1 and 2 (see also a direct derivation of the scaling behavior for $q = 2$ in Appendix IV). Again, exponents $\varphi(q)$ (resp., $\varphi(qH)$) can be estimated from linear regressions in $\log \mathbb{E}[(\delta_\tau A/\tau)^q]$ versus $m(\mathcal{C}_\tau)$ (resp., $\log \mathbb{E}[(\delta_\tau V/\tau^H)^q]$ versus $m(\mathcal{C}_\tau)$) diagrams: see Fig. 6.

In this precise case, departures from power-law behaviors are even directly (slightly) visible without correcting the τ^q term in $\mathbb{E}[\delta_\tau A^q]$. For instance, Fig. 7 shows that $\log \mathbb{E}\left[\delta_\tau A^2\right]$ is close to but does not exactly fit a linear function of $\log\tau$ for $\tau < 1$. Note that the importance of this departure from a power-law behavior depends on the precise order q of the considered moment $\mathbb{E}[\delta_\tau A^q]$. Indeed, this effect is proportional to $\varphi(q)m(\mathcal{C}_\tau)$ in a log-log diagram. For instance, when $q = 1$, no departure will ever be observed since $\varphi(1) = 0$ by definition.

At this point, let us emphasize that, to our knowledge, this is the first example of a multiplicative cascade displaying controlled non-power-law behaviors up to a large range of scales (four decades on Fig. 4).

VI. Algorithms for Practical Synthesis

This section is devoted to the key points entering the practical algorithms aiming at the simulation of the IDC noise, motion, and random walk. The corresponding MATLAB routines have been developed and used to produce the illustrations of Section V. They are freely available and documented upon request from the authors. Despite theoretical similarities, there are important practical differences between the specific case of CPCs and the general case of IDCs.[8] They are presented separately in Sections VI-A and -B for the synthesis of the IDC noise and IDC motion. Then Section VI-C explains how to obtain an IDC random walk $V_{H,r}$ from an IDC motion $A_r(t)$.

Though the defined process are continuous-time processes, algorithms output samples with a uniform sampling rate $\Delta t \ll 1$. Let $[0,\ T]$, with $T > 1$ denote the interval over which processes are to be produced. With our definitions, the scaling properties are prescribed in the range of scales $r \leq \tau \leq 1$. Using the sampling period as a time reference, the characteristic scales of the constructions are $r/\Delta t$, $1/\Delta t$, and $T/\Delta t$.

[8]Recall that not all infinitely divisible distributions are compound Poisson distributions.

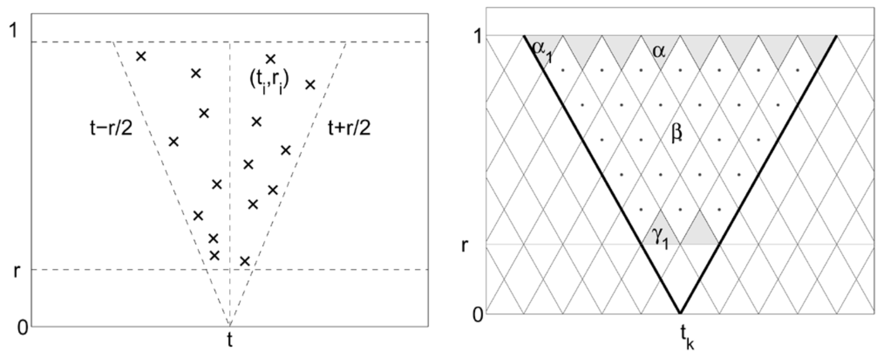

Fig. 8. (Left) CPCs are built on a Poisson point process $\{(t_i, r_i)\}$ and random i.i.d. multipliers W_i. (Right) IDCs necessitate an adapted discretization of the time–scale plane: schema of the different triangular (▼, ▲), and lozenge (⧫) subsets that contribute to $Q_r(t_k)$. This schematic vision is translated in the form of the matrix $\Psi(t_k)$.

A. Simulation of CPCs

As explained in Section II-D, CPCs are built from two ingredients: a planar Poisson point process $\{(t_i, r_i)\}$ with density $dm(t,r)$ in $\mathcal{P}^+$ and i.i.d. random multipliers W_i with $\log W_i$ distributed by F. The planar point process provides us with a natural sampling of the time–scale plane (see Fig. 8 (left)), which makes things simple. Let the trapezoid

$$\Theta = \{(t', r') : r \le r' \le 1, -r'/2 \le t' \le T + r'/2\}.$$

The synthesis algorithm consists of the following steps for given resolution r and duration T.

1) Determine the number N_p of points (and multipliers) that will be used to compute $Q_r(t)$ in the interval $[0, T]$: it is a Poisson random variable with parameter $m(\Theta)$.
2) Select N_p random points (t_i, r_i) located in the trapezoid Θ, according to density $dm(t,r)$.[9]
3) Select N_p i.i.d. random multipliers W_i such that $\log W_i$ are distributed by F.
4) For each time position $t \in \{t_k = k\Delta t, 0 \le k \le T/\Delta t\}$, set

$$Q_r(t) = \exp[(1 - \mathbb{E}W) m(\mathcal{C}_r(t))] \cdot \prod_{(t_i, r_i) \in \mathcal{C}_r(t)} W_i.$$

5) The approximate version $A_r(t)$ of an IDC motion $A(t)$ is obtained as the discrete time integral of $Q_r(t)$

$$A_r(t) = \sum_{0 \le k \le t/\Delta t} Q_r(t_k) \cdot \Delta t. \qquad (62)$$

A key feature of this algorithm is that it is easy to implement and has a low computational cost. Little modification is necessary to get a causal version, in the spirit of the recursive algorithm proposed for non-CPC infinitely divisible cascades below.

B. Simulation of Infinitely Divisible Cascades

Let us now turn to the simulation of (noncompound Poisson) infinitely divisible cascades such as, e.g., the normal cascade. The construction is no longer based on a discrete random point process but rather on a continuous and independently scattered random measure M on the time–scale plane $\mathcal{P}^+$: no natural sampling appears. A relevant sampling of the $\mathcal{P}^+$ must therefore be chosen, immediately rising the issues of computational cost and available memory. To tackle this problem, a causal recursive algorithm is proposed in order to simulate $Q_r(t)$ for each $\{t_k = k\Delta t, k \in \mathbb{N}\}$.

Fig. 8 (right) gives an intuitive picture of our algorithm. With little restriction, the sampling period is chosen as the inverse of an integer $\Delta t = 1/\lambda_t$ while the resolution r is chosen as $r = \lambda_r \Delta t$, with $\lambda_t, \lambda_r \in \mathbb{N}^*$. Therefore, a natural discretization of the plane $\mathcal{P}^+$ appears as a combination of downwards triangles ▼ (with random measure denoted by α), lozenges ⧫ (with measure denoted by β), and upwards triangles ▲ (with measure denoted by γ)—see Fig. 8 (right). At each time t_k, the terms that contribute to $Q_r(t_k)$ can be gathered in the following triangular matrix $\Psi(t_k)$:

$$\Psi(t_k) = \begin{bmatrix} \alpha_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \alpha_{\lambda_t} \\ & \beta_{1,1} & \cdots & \cdots & \cdots & \cdots & \beta_{1,\nu_1} \\ & & \ddots & & & & \vdots \\ & \mathbf{0} & & \beta_{\nu_0,\nu_0} & \cdots & \cdots & \beta_{\nu_0,\nu_1} \\ & & & & \gamma_1 & \cdots & \gamma_{\lambda_r} \\ & & & & & \mathbf{0}_{\lambda_r - 1} & \end{bmatrix} \qquad (63)$$

where $\nu_0 = \lambda_t - \lambda_r - 1$ and $\nu_1 = \nu_0 + \lambda_r = \lambda_t - 1 > \nu_0$; $\mathbf{0}_{\lambda_r - 1}$ denotes a zero square matrix of size $\lambda_r - 1$. $\Psi(t_k)$ is a $\lambda_t \times \lambda_t$ square matrix. We denote by $D(t_k)$ its diagonal and $C_{\lambda_t}(t_k)$ its last column

$$\begin{cases} C_{\lambda_t}(t_k) = (\alpha_{\lambda_t}, \beta_{1,\nu_1}, \ldots, \beta_{\nu_0,\nu_1}, \gamma_{\lambda_r}, 0_{\lambda_r - 1}) \\ D(t_k) = (\alpha_1, \beta_{1,1}, \ldots, \beta_{\nu_0,\nu_0}, \gamma_1, 0_{\lambda_r - 1}). \end{cases}$$

For the given infinitely divisible distribution G (with moment generating function $\tilde{G}(q) = e^{-\rho(q)}$) and control measure $dm(t,r)$, the simulation consists of the following steps.

1) Compute $m(▼)$ and $m(▲)$ as well as the various $m(⧫)$ depending on the position of the lozenge;
2) Simulate the $\alpha = M(▼)$, random variables distributed according to $G_{m(▼)}$ (with moment generating function $\tilde{G}(q) = e^{-\rho(q) m(▼)}$, see Appendix I); do the same with $\beta = M(⧫)$ and $\gamma = M(▲)$ distributed by $G_{m(⧫)}$ and $G_{m(▲)}$, respectively.

[9]The nonuniform distribution $rg(r)$ of the r_i is achieved by a change of variable from a uniformly distributed random variable.

3) Initialize $\Psi(t=0)$, and set

$$Q_r(0) = \exp(\rho(1)m(\mathcal{C}_r)) \exp \sum_{i,j} \Psi_{i,j}(0).$$

4) Recursively obtain $\Psi(t_{k+1})$ from $\Psi(t_k)$ through the following procedure:
   a) eliminate diagonal $D(t_k)$,
   b) translate all coefficients of $\Psi(t_k)$ one column to the left,
   c) simulate a new last column $C_{\lambda_t}(t_{k+1})$, and insert it to form $\Psi(t_{k+1})$,
   d) then get $Q_r(t_{k+1})$ using

$$Q_r(t_{k+1}) = Q_r(t_k) \cdot \exp\left(\sum_i C_{\lambda_t,i}(t_{k+1}) - \sum_i D_i(t_k)\right).$$

5) Repeat until $t_k = T$ (that is as long as needed, with no limitation on the value of T).
6) IDC motion A_r is obtained by simple integration (as in (62))

$$A_r(t) = \sum_{0 \le k \le t/\Delta t} Q_r(t_k) \cdot \Delta t. \tag{64}$$

The matrix $\Psi(t_k)$ plays the role of a "memory" of the process. In a way, it *propagates* the correlation structure of process Q_r. This method of simulation results in a causal construction. The adaptation of this algorithm to CPCs is left to the reader.

C. Simulation of an IDC Random Walk

Once an IDC motion $A_r(t)$ has been simulated, one obtains $V_{H,r}(t)$ in two steps.

1) Simulate a fractional Brownian motion B_H with Hurst parameter H thanks to the fast circulating matrix method [43], [44]. This fractional Brownian motion is oversampled by a factor p compared to A_r, i.e., it is synthetized on a grid t'_j with a sampling rate $\Delta t' = \Delta t/p$ (for instance, $p = 16$).
2) Set $V_{H,r}(t_k) = B_H(t'_k)$ where t'_k is such that

$$|t'_k - A_r(t_k)| = \inf_{t'_j} |t'_j - A_r(t_k)|.$$

The processes Q_r, A_r, and $V_{H,r}$ shown in previous sections have been produced with the algorithms described here.

VII. Conclusion and Perspectives

In the present work, we proposed the definitions of continuous-time processes that exhibit controlled continuous multiscaling behavior. Mostly, scaling laws are continuous through the scales and possible departures from a pure power-law behavior are taken into account. To our knowledge and despite some limitations, IDC processes are the first continuous multiplicative cascades displaying controlled non-power-law scaling behaviors. Moreover, algorithms for practical synthesis are given. MATLAB functions as well as a companion paper [27] that puts the emphasis on more applied aspects are available from our web pages (see www.isima.fr/~chainais; http://perso.ens-lyon.fr/patrice.abry).

The theoretical study of the scaling properties of these processes brought better understanding and new intuitions about the subtle interplay between cascading mechanisms and scaling phenomena. Aiming at a better localized control in the time–scale plane, we are currently elaborating a variation on this construction of the form

$$Q_r(t) = \frac{\exp \int f\left(\frac{t-t'}{r'}\right) dM(t',r')}{\mathbb{E} \exp \int f\left(\frac{t-t'}{r'}\right) dM(t',r')} \tag{65}$$

where $f(t)$ is some bounded support function. Note that Definition 1 is recovered for the choice $f = \mathbf{1}_{[-1/2,1/2]}$. Potential improvements of such a generalized formulation are under study.

In practice, IDC processes could relevantly and efficiently replace the usual binomial cascades which remain the most commonly used tools in applications. We put the emphasis on the fact that the ability to account for departures from exact power laws is a major practical improvement for the modeling of real empirical data.

The use of such processes to calibrate analysis and estimation tools should be of major benefit. We are currently investigating the performances of the most commonly used analysis tools thanks to those reference processes [45]. We are also designing new estimators for non-power-law scaling.

Applications to hydrodynamic turbulence and to computer network traffic are under development.

Appendix I
Infinitely Divisible Distributions

Let us recall some basics on infinitely divisible probability distributions or laws. We denote the set of strictly positive integers by $\mathbb{N}^*$.

Definition: A distribution G is called infinitely divisible if for all $n \in \mathbb{N}^*$ there exists a distribution G_n such that G equals the n-fold convolution of G_n with itself, denoted as $(G_n)^{n\star}$.

In other words, the distribution of a random variable S is infinitely divisible if and only if for all $n \in \mathbb{N}^*$ the variable S can be written in law as the sum of n i.i.d. variables

$$S \overset{\text{law}}{=} Y_{1,n} + \cdots + Y_{n,n}.$$

Clearly, the distribution of $Y_{i,n}$ is G_n from the above definition. Again in different words, a distribution G with characteristic function $e^{-\psi}$ is infinitely divisible if and only if for all $n \in \mathbb{N}^* e^{-\psi/n}$ is again a characteristic function. Moreover, one has the following.

Theorem (see Feller [32, p. 432]): Every characteristic function $\mathbb{E}[e^{iqX}]$ of an infinitely divisible law is necessarily of the form $\exp(-\psi(\cdot))$. If $e^{-\psi}$ is the characteristic function of an infinitely divisible distribution G, then for all $s > 0$, $e^{-s\psi}$ is the characteristic function of an infinitely divisible distribution G_s.

Note that the same theorem can be written for the moment generating function $\tilde{G}(q) = \mathbb{E}[e^{qX}] = e^{-\rho(q)}$ for values of q such that it exists.

APPENDIX II
INDEPENDENTLY SCATTERED RANDOM MEASURES

To introduce random measures on the upper half plane $\mathcal{P}^+ = \{(t,r) : t \in \mathbb{R}, r > 0\}$ the following notion is useful.

Definition: An independently scattered (Borel) random measure M on $\mathcal{P}^+$ is a measure-valued process defined on the Borel sets of $\mathcal{P}^+$ such that for all disjoint sets $\mathcal{E}_1$ and $\mathcal{E}_2$

- $M(\mathcal{E}_1)$ and $M(\mathcal{E}_2)$ are independent random variables,
- $M(\mathcal{E}_1 \cup \mathcal{E}_2) = M(\mathcal{E}_1) + M(\mathcal{E}_2)$.

The additivity property makes it natural to construct such random measures in an infinitely divisible framework. Leveraging Feller's theorem from the previous section and following Rajput and Rosinski [46] and Samorodnitsky and Taqqu [34] one defines the following.

Definition: Let G be an infinitely divisible distribution with moment generating function $\tilde{G} = e^{-\rho}$. Let $dm(t,r)$ be a positive deterministic measure on $\mathcal{P}^+$. Then, a measure M with the two properties listed below is called *random measure with control measure* m and *generator* G

- M is an independently scattered Borel measure on $\mathcal{P}^+$;
- for any Borel set $\mathcal{E}$ of $\mathcal{P}^+$ the random variable $M(\mathcal{E})$ distributed as $G_{m(\mathcal{E})}$, i.e.,

$$\mathbb{E}[\exp[qM(\mathcal{E})]] = \exp[-\rho(q)m(\mathcal{E})]. \qquad (66)$$

If the choice of the infinitely divisible law G is obvious from the context, we may call M simply *infinitely divisible measure with control measure* m. Prominent examples are given in the text, such as normal or compound Poisson distributions for instance (see Section II-D).

APPENDIX III
REMAINING PROOFS OF SECTION III-C

Lemma 6: Let $0 < r \le b^n < 1$. Then $A_r^{(n)}$ is independent of Q_{b^n}, thus also of $Q_{b^n}^{b^{n-1}}$ and

$$A_{r/b^{n-1}}^{(n-1)}(t/b^{n-1}) = \frac{b}{b^n}\int_0^t Q_{b^n}^{b^{n-1}}(s) Q_r^{b^n}(s)ds$$
$$= b\int_0^t Q_{b^n}^{b^{n-1}}(s) d\left[A_{r/b^n}^{(n)}\left(\frac{s}{b^n}\right)\right]. \qquad (67)$$

Proof: Simply plug the recursion (25) into the definition (37) of $A^{(n)}$. □

Lemma 7: Fix $q > 0$. Let $0 < r \le b^n < 1$ and $t > 0$. Then

$$\mathbb{E}\left|A_{r/b^{n-1}}^{(n-1)}(t/b^{n-1})\right|^q = b^q \cdot \mathbb{E}\left[Q_{b^n}^{b^{n-1}}(0)^q\right]$$
$$\cdot \mathbb{E}\left[A_{r/b^n}^{(n)}(t/b^n)^q\right] \cdot \left(1 + \varepsilon_r^{(n)}(t/b^n)\right). \qquad (68)$$

The error term $\varepsilon_r^{(n)}$ is bounded as

$$\left|\varepsilon_r^{(n)}(s)\right| \le \Delta_{b,q}^{(n)}(s).$$

Proof: We will be using the fact[10] (see [47]) that for any positive measure μ and any positive q

$$\left|\left(\int_I x(s)d\mu(s)\right)^q - C\right| \le \sup_{s\in I}|x(s)^q\mu(I)^q - C|. \qquad (69)$$

Applying Lemma 6, then (69) with $I = [0,t]$ to the measure μ induced by $A_{r/b^n}^{(n)}(\cdot/b^n)$, and finally using (12) and (35), we may write the following:

$$\left|A_{r/b^{n-1}}^{(n-1)}(t/b^{n-1})^q - b^q Q_{b^n}^{b^{n-1}}(0)^q A_{r/b^n}^{(n)}(t/b^n)^q\right|$$
$$= \left|\left(b\int_0^t Q_{b^n}^{b^{n-1}}(s)d\left[A_{r/b^n}^{(n)}(s/b^n)\right]\right)^q \right.$$
$$\left. - b^q \cdot Q_{b^n}^{b^{n-1}}(0)^q \cdot A_{r/b^n}^{(n)}(t/b^n)^q\right|$$
$$\le \sup_{0\le s\le t}\left|b^q Q_{b^n}^{b^{n-1}}(s)^q A_{r/b^n}^{(n)}(t/b^n)^q \right.$$
$$\left. - b^q Q_{b^n}^{b^{n-1}}(0)^q A_{r/b^n}^{(n)}(t/b^n)^q\right|$$
$$= b^q \cdot A_{r/b^n}^{(n)}(t/b^n)^q$$
$$\cdot \sup_{0\le s\le t}\left|Q_{b^n}^{b^{n-1}}(s)^q - Q_{b^n}^{b^{n-1}}(0)^q\right|. \qquad (70)$$

Using $|\mathbb{E}X - \mathbb{E}Y| \le \mathbb{E}|X - Y|$ and the definition of $\Delta_{b,q}^{(n)}$ the claim follows. □

Proof of Theorem 1

In order to establish Theorem 1, we would like to iterate the recursion (68) n times keeping b fixed. Thus, we will apply the recursion successively with t/b^k to the cascades $A_r^{(k)}$ introduced in (37), for $k = 1, \ldots, n$. According to Lemma 7 we find (provided $r < b^n$)

$$\mathbb{E}A_r(t)^q = \mathbb{E}\left[A_{r/b^n}^{(n)}(t/b^n)^q\right]$$
$$\cdot \prod_{k=1}^n b^q \cdot \mathbb{E}\left[Q_{b^k}^{b^{k-1}}(0)^q\right]\left(1 + \varepsilon_r^{(k)}(t/b^k)\right)$$
$$= b^{nq}\mathbb{E}\left[Q_{b^n}(0)^q\right] \cdot \mathbb{E}\left[A_{r/b^n}^{(n)}(t/b^n)^q\right]$$
$$\cdot \prod_{k=1}^n \left(1 + \varepsilon_r^{(k)}(t/b^k)\right). \qquad (71)$$

Here, we used mutual independence of the $Q_{b^k}^{b^{k-1}}$ (25) to collect the moments.

Let us first consider the case $t = b^n$. Fixing n, letting $r \to 0$, and using $\mathcal{L}^q$-convergence yields

$$\mathbb{E}A(b^n)^q = b^{nq}\mathbb{E}\left[Q_{b^n}(0)^q\right] \cdot \mathbb{E}\left[A^{(n)}(1)^q\right] \cdot$$
$$\cdot \prod_{k=1}^n \left(1 + \varepsilon^{(k)}(b^{n-k})\right). \qquad (72)$$

[10]Indeed, since $\int_I x(s)d\mu(s) \le \sup_s(x(s)\mu(I))$ we have $(\int_I x(s)d\mu(s))^q \le \sup_{s\in I}(x(s)^q\mu(I)^q)$. Now subtract C from both sides. Similarly, $C - (\int_I x(s)d\mu(s))^q \le C - \inf_s(x(s)^q\mu(I)^q) = \sup(C - x(s)^q\mu(I)^q)$.

Here, the error terms are obtained by taking limits in (68). Listing them in reverse order for convenience, they read as

$$\varepsilon^{(n-i)}(b^i) = \frac{\mathbb{E}[A^{(n-i-1)}(b^{i+1})^q]}{b^q \mathbb{E}\left[Q_{b^{n-i}}^{b^{n-i-1}}(0)^q\right] \mathbb{E}[A^{(n-i)}(b^i)^q]} - 1. \quad (73)$$

Note that for fixed i, each term $\varepsilon^{(n-i)}(b^i)$ converges as $(n \to \infty)$. Indeed, the finite-dimensional distributions of $A^{(n)}$ depend only and continuously on $m^{(n)}(\mathcal{C}_r(t) \cap \mathcal{C}_r(s))$ [23]; but $m^{(n)}$ converges by assumption. In particular, if $m^{(n)}$ converges to the trivial zero measure, then $Q_{b^{n-i}}^{b^{n-i-1}}$ and $A^{(n)}$ converge in distribution to the constant 1.

In addition, Lemma 7 $|\varepsilon^{(n-i)}(b^i)| \leq C(b^\nu)^i$. Since the sum $\sum_i (b^\nu)^i$ converges absolutely, the product [5] $\prod_{i=0}^{n-1}(1+\varepsilon^{(n-i)}(b^i))$ will converge by bounded convergence to a finite, nonzero limit which can be consumed in the constants.

Similarly, the terms $\mathbb{E}[A^{(n)}(1)^q]$ converge and can be consumed in the constants.

Finally, the bounds have to be extended for all $t \in [0,1]$. Since $A_r^q(\cdot)$ is a nondecreasing process, it is an easy exercise to show that a correction factor for the constant bound large enough is $b^q \sup_n \mathbb{E}[Q_{b^n}^{b^{n-1}}(0)^q]$. However, similarly as before

$$\begin{aligned}\mathbb{E}\left[Q_{b^n}^{b^{n-1}}(0)^q\right] &= \exp\left[-\varphi(q) m\left(\mathcal{C}_{b^n}^{b^{n-1}}(0)\right)\right] \\ &= \exp\left[-\varphi(q) m^{(n-1)}(\mathcal{C}_b(0))\right]\end{aligned}$$

converges in n, this factor is bounded. □

Proof of Corollary 2

We establish Corollary 2 via three lemmas. The first result is general and simplifies $\Delta_{b,q}^{(n)}$ by separating independent from dependent parts of $M(\mathcal{C}_{b^n}^{b^{n-1}}(u))$ and $M(\mathcal{C}_{b^n}^{b^{n-1}}(0))$. To this end we introduce the following parallelepiped as subsets of the time–scale strip (see Fig. 9 for $n=1$):

$$\begin{aligned}\mathcal{L}(u,v) &= \{(s,r) : b^n \leq r \leq b^{n-1}, -r+u \leq s < -r+v\} \\ \mathcal{B}(t) &= \mathcal{C}_b(t) \cap \mathcal{C}_b(0) \\ &= \{(s,r) : b^n \leq r \leq b^{n-1}, -r+t \leq s \leq r\} \\ \mathcal{R}(u,v) &= \{(s,r) : b^n \leq r \leq b^{n-1}, r+u \leq s < r+v\}. \quad (74)\end{aligned}$$

Lemma 8: For $0 \leq t \leq 1$

$$\begin{aligned}\Delta_{b,q}^{(n)}(t) &\leq \exp[\rho(q) m(\mathcal{L}(0,tb^n))] \\ &\times \mathbb{E}\left[e^{qM(\mathcal{L}(0,tb^n))} \sup_{0\leq u\leq tb^n}\left|\frac{e^{qM(\mathcal{R}(0,u))}}{e^{qM(\mathcal{L}(0,u))}} - 1\right|\right]. \quad (75)\end{aligned}$$

Proof: First, we cancel the normalization terms of Q which appear in $\Delta_{b,q}^{(n)}$: see (76) at the bottom of the page.

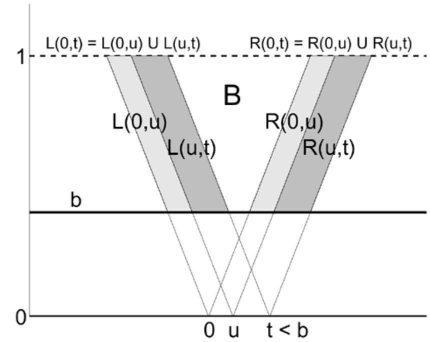

Fig. 9. Definition of $\mathcal{L}$, $\mathcal{B}$, and $\mathcal{R}$.

Next, checking the constraints on the variable s in (74) one verifies quickly the following decomposition of a cone $\mathcal{C}_{b^n}^{b^{n-1}}(u)$ into *disjoint* sets which is valid for $u \in [0, tb^n]$ and for $t \leq 1$ (see Fig. 9 for $n=1$):

$$\mathcal{C}_{b^n}^{b^{n-1}}(u) = \mathcal{L}(u, tb^n) \cup \mathcal{B}(tb^n) \cup \mathcal{R}(0,u). \quad (77)$$

As a particular case, we have $\mathcal{C}_{b^n}^{b^{n-1}}(0) = \mathcal{L}(0,tb^n) \cup \mathcal{B}(tb^n)$. Thus,

$$\begin{aligned}\Delta_{b,q}^{(n)}(t) = &\frac{\mathbb{E}[\exp[qM(\mathcal{B}(tb^n))]]}{\mathbb{E}\left[\exp\left[qM\left(\mathcal{C}_{b^n}^{b^{n-1}}(0)\right)\right]\right]} \\ &\times \mathbb{E}\Big[\sup_{0\leq u\leq tb^n}\big|\exp\left[qM(\mathcal{L}(u,tb^n) \cup \mathcal{R}(0,u))\right] \\ &\quad - \exp[qM(\mathcal{L}(0,tb^n))]\big|\Big]. \quad (78)\end{aligned}$$

Here, we used that the term $e^{qM(\mathcal{B}(tb^n))}$ is statistically independent of the other terms in the enumerator. We note that

$$\begin{aligned}\frac{\mathbb{E}[e^{qM(\mathcal{B})}]}{\mathbb{E}\left[e^{qM(\mathcal{C}_{b^n}^{b^{n-1}}(0))}\right]} &= \exp\left[-\rho(q)(m(\mathcal{B}) - m(\mathcal{C}_{b^n}^{b^{n-1}}(0)))\right] \\ &= \exp\left[\rho(q) m(\mathcal{L}(0,tb^n))\right]. \quad (79)\end{aligned}$$

Finally, since $\mathcal{L}(u,v) \cup \mathcal{L}(v,w) = \mathcal{L}(u,w)$ with disjoint union whenever $u \leq v \leq w$, we find (75). □

It remains to bound the second term in (75) which we achieve in the special cases of CPC and log-normal cascades. Now the idea is to show that with t very small and thus u small, the control measures $m(\mathcal{R}(0,u))$ and $m(\mathcal{L}(0,u))$ are very small, thus, the corresponding random variables are small with high probability and thus $e^{qM(\mathcal{R}(0,u))}$ and $e^{qM(\mathcal{L}(0,u))}$ are both close to 1. Thus, their quotient is close to one and the contribution to the last term in (75) is small with large probability.

Compound Poisson Cascades. As a matter of fact, that quotient is exactly equal to 1 with large probability in the CPC case which is the main ingredient to the next result. The log-normal case is somewhat more intricate.

$$\Delta_{b,q}^{(n)}(t) = \frac{\mathbb{E}\sup_{0\leq u\leq tb^n}\left|\exp\left[qM\left(\mathcal{C}_{b^n}^{b^{n-1}}(u)\right)\right] - \exp\left[qM\left(\mathcal{C}_{b^n}^{b^{n-1}}(0)\right)\right]\right|}{\mathbb{E}\left[\exp\left[qM\left(\mathcal{C}_{b^n}^{b^{n-1}}(0)\right)\right]\right]}. \quad (76)$$

Lemma 9: Fix $q > 0$ and $n \in \mathbb{N}$. Let $0 < t \leq 1$. If the weights W of a CPC have finite qth moments, then there exist finite constants $C^{(n)}$ (see (75) and (85)) such that

$$\Delta_{b,q}^{(n)}(t) \leq t \cdot C^{(n)}. \tag{80}$$

Assume in addition that

$$\int_b^1 g^{(n)}(r)dr = b^n \int_{b^{n+1}}^{b^n} g(s)ds$$

is bounded. Then $C^{(n)} \leq C$ for some constant C.

Proof: Since $e^{qM(\mathcal{R}(0,u))} = e^{qM(\mathcal{L}(0,u))}$ with high probability, the following crude bound will suffice to bound the error term. To this end, denote by $N(E)$ the number of Poisson points falling in some set E and define

$$L(E) := \prod_{(t_i,r_i)\in E} \left(W_i^q + 1\right). \tag{81}$$

By standard computation (L is in essence a CPC but with new weights, compare Example 4) we find, using that $m(E) = \mathbb{E}[N(E)]$

$$\begin{aligned}\mathbb{E}[L(E)] &= \sum_{k=0}^{\infty} \frac{m(E)^k}{k!} e^{-m(E)} \mathbb{E}[W^q + 1]^k \\ &= \exp[\mathbb{E}[W^q] m(E)]\end{aligned} \tag{82}$$

which is finite since W has a finite qth moment. Consider the set

$$\mathcal{E} := \mathcal{L}(0, tb^n) \cup \mathcal{R}(0, tb^n). \tag{83}$$

Since $\mathcal{E}$ contains all Poisson points which may possibly appear in the supremum in (75) (see also (78)) and since $W^q + 1 \geq \max(1, W^q)$, we find immediately using $|a-b| \leq |a|+|b|$, (84) at the bottom of the page; in particular, $\Delta_{b,q}^{(n)}(t) < \infty$. To advance to a more accurate estimate, let us note that the supremum in (84) actually vanishes whenever no Poisson points fall in $\mathcal{E}$, i.e., whenever $N(\mathcal{E}) = 0$. But the probability of this happening is $P[N(\mathcal{E}) = 0] = \exp(-m(\mathcal{E}))$. Since $1 - e^{-a} \leq a$

$$\begin{aligned}&\mathbb{E}\left[e^{qM(\mathcal{L}(0,tb^n))} \sup_{0\leq u\leq tb^n} \left|\frac{e^{qM(\mathcal{R}(0,u))}}{e^{qM(\mathcal{L}(0,u))}} - 1\right|\right] \\ &\quad\leq \Pr[N(\mathcal{E}) \neq 0] \cdot \mathbb{E}[2L(\mathcal{E})] \\ &\quad= m(\mathcal{E}) \cdot 2\exp[\mathbb{E}[W^q] m(\mathcal{E})].\end{aligned} \tag{85}$$

Since

$$m(\mathcal{E}) = 2m(\mathcal{L}(0, tb^n)) = 2tb^n \int_{b^n}^{b^{n-1}} g(r)dr \tag{86}$$

the claims follow with lemma 8. □

Normal Cascades. Finally, Theorem 1 applies to any normal cascade as we show now. This will complete the proof of Corollary 2.

Lemma 10: Fix $q > 0$. Let $0 < t \leq 1$. For any log-normal IDC there exist finite constants $C^{(n)}$ such that

$$\Delta_{b,q}^{(n)}(t) \leq \sqrt{t} \cdot C^{(n)}. \tag{87}$$

Assume in addition that $\int_b^1 g^{(n)}(r)dr$ (38) are bounded for all n, then $C^{(n)}$ remains bounded as $n \to \infty$.

Proof: Let us recall that $\rho(q) = -q\mu - q^2\sigma^2/2$, i.e., $M(E)$ is $\mathcal{N}(m(E)\mu, m(E)\sigma^2)$.

Step 1: Consider the following processes:

$$\begin{cases} X_s := qM(\mathcal{L}(0,s)) - qm(\mathcal{L}(0,s))\mu \\ Y_s := qM(\mathcal{R}(0,s)) - qm(\mathcal{R}(0,s))\mu. \end{cases} \tag{88}$$

Both are Gaussian processes with independent stationary increments of zero mean; thus, they form Brownian motions which are unique up to a multiplicative scalar which can be set through the variance at time $s = 1$. In particular, they are thus statistically self-similar, i.e., X_{as} and $\sqrt{a}X_s$ are equal in the sense of finite-dimensional distributions. Note in addition that $\{X_s\}_{s\leq tb^n}$ and $\{Y_s\}_{s\leq tb^n}$ are independent since M is randomly scattered.

Set $S_X(s) = \sup_{0\leq u\leq t} |X_u|$ and $S_Y(s)$ similarly. Considering continuous versions of the motion only we are lead to

$$S_X(s) = \sup_{0\leq u\leq s} |X_u| \stackrel{\text{fdd}}{=} \sqrt{s} \cdot \sup_{0\leq u\leq 1} |X_u| = \sqrt{s} \cdot S_X(1) \tag{89}$$

with equality in the sense of finite-dimensional distributions. From Leadbetter [48, Lemma 12.2.1, p. 219] we borrow the following fact: Since X_s and $-X_s$ are normal processes with $\mathbb{E}[X_s] = 0$ for all s, $\mathbb{E}\left[X_0^2\right] = 0$ and $\mathbb{E}\left[(X_t - X_s)^2\right] = \xi^2|t-s|$ we have

$$P\left[\sup_{0\leq s\leq 1} |X_s| > x\right] = P[S_X(1) > x] \leq 8\exp\left(-\frac{c}{\xi^2}x^2\right) \tag{90}$$

where c is a real constant which does not depend on any statistics of X_s and where

$$\begin{aligned}\xi^2 := \mathbb{E}[X(1)^2] &= q^2 \text{var}[M(\mathcal{L}(0,1))^2] \\ &= q^2\sigma^2 m(\mathcal{L}(0,1)).\end{aligned} \tag{91}$$

The same bound (90) with the same ξ holds for Y_s since $\mathbb{E}[Y(1)^2] = \sigma^2 m(\mathcal{R}(0,1)) = \xi^2$.

Step 2: For simplicity of notation, we assume here $n = 0$; more generally, every t has to be replaced by tb^n. Setting

$$\begin{aligned}I := \mathbb{E}&\left[e^{X_t} \cdot \sup_{0\leq s\leq t} |e^{Y_s - X_s} - 1| \middle| S_X(t) \geq 1 \text{ or } S_Y(t) \geq 1\right] \\ &\times P[S_X(t) \geq 1 \text{ or } S_Y(t) \geq 1]\end{aligned} \tag{92}$$

$$\begin{aligned}&\sup_{0\leq u\leq tb^n} \left|\exp[qM(\mathcal{L}(u,tb^n) \cup \mathcal{R}(0,u))] - \exp[qM(\mathcal{L}(0,tb^n))]\right| \\ &\quad= \sup_{0\leq u\leq tb^n} \left|\prod_{\mathcal{L}(u,tb^n)\cup\mathcal{R}(0,u)} W_i - \prod_{\mathcal{L}(0,tb^n)} W_i\right| \\ &\quad\leq 2L(\mathcal{E})\end{aligned} \tag{84}$$

$$\mathbb{I} := \mathbb{E}\left[e^{X_t} \cdot \sup_{0\le s\le t} |e^{Y_s - X_s} - 1| \Big| S_X(t) < 1, S_Y(t) < 1\right] \times P[S_X(t) < 1, S_Y(t) < 1] \quad (93)$$

we have, since $m(\mathcal{R}(0,s)) = m(\mathcal{L}(0,s))$

$$\mathbb{E}\left[e^{qM(\mathcal{L}(0,t))} \sup_{0\le s\le t}\left|e^{qM(\mathcal{R}(0,s)) - qM(\mathcal{L}(0,s))} - 1\right|\right] = e^{qm(\mathcal{L}(0,t))\mu}(I + \mathbb{I}). \quad (94)$$

To estimate I and $\mathbb{I}$ it useful to observe that for all $a \in \mathbb{R}$ we have $|e^a - 1| \le e^{|a|} - 1$, and thus

$$|e^{(y-x)} - 1| \le e^{|y-x|} - 1 \le \begin{cases} e^{|y|+|x|}, & \text{for all } x, y \\ 5(|x| + |y|), & \text{if } |x| \le 1,\ |y| \le 1. \end{cases}$$

Indeed, for $a < 0$ we have $|e^a - 1| = 1 - e^a \le e^{-a} - 1$ since $(e^{a/2} - e^{-a/2})^2 \ge 0$. The constant 5 could be slightly improved.

First, to estimate I we define the events

$$E_{k,n} := \{k > S_X(t) \ge k - 1,\ n > S_Y(t) \ge n - 1\}$$

and

$$F_{1,1} := \{S_X(t) > 1 \text{ or } S_Y(t) > 1\}.$$

Conditioned on $E_{k,n}$ the following bound holds for all $0 \le s \le t$:

$$e^{X_t} \cdot |\exp(Y_s - X_s) - 1| \le e^k(e^n e^k) = e^{2k+n}.$$

Taking the supremum over s and using self-similarity (89) and (90), together with the independence of X_s and Y_s for $0 \le s \le b$ we find that I is bounded from above by the quantity

$$\sum_{k,n=1}^{\infty} \mathbb{E}\left[e^{X_t} \cdot \sup_{0\le s\le t} |e^{Y_s - X_s} - 1| \Big| E_{k,n}, F_{1,1}\right] \cdot P[E_{k,n}, F_{1,1}] \le \sum_{(k,n)\ne(1,1)} e^{n+2k} \cdot 64 \cdot e^{-\frac{c}{\xi^2}\frac{(k-1)^2}{t}} \cdot e^{-\frac{c}{\xi^2}\frac{(n-1)^2}{t}} < \infty. \quad (95)$$

Elementary estimates using $\exp(-u) \le 1/u$ show that I is not only finite (note that $\mathbb{I}$ is trivially finite) but $O(t)$ as $t \to 0$ with a prefactor that can be made arbitrarily close to $64\xi^2/c$. As a matter of fact, I is $O(t^k)$ for any k.

Second, using that $\mathbb{E}[V|V < z]P[V < z] \le \mathbb{E}[V]$ for any positive random variable V and (89) we find

$$\begin{aligned}\mathbb{I} &\le \mathbb{E}\left[e \cdot 5 \sup_{0\le s\le t}(|Y_s| + |X_s|) \Big| S_X(t) < 1, S_Y(t) < 1\right] \times P[S_X(t) < 1,\ S_Y(t) < 1] \\ &= 5e \cdot \mathbb{E}[S_X(t) + S_Y(t)] \\ &= \sqrt{t} \cdot 10e \cdot \mathbb{E}[S_X(1)] \\ &= \sqrt{t} \cdot \xi \cdot 10e \cdot \mathbb{E}[S_B(1)].\end{aligned}$$

Here, $B(s)$ denotes normalized Brownian motion with $\mathbb{E}[B^2(1)] = 1$; $\mathbb{E}[S_B(1)]$ is a known constant number.

Step 3: In summary: using $\rho(q) = -q\mu - q^2\sigma^2/2$ and Lemma 8, there are constants c_1 and c_2 independent of any parameters such that

$$\Delta_{b,q}^{(n)}(t) \le \left(c_1\xi^2 tb^n + c_2\xi\sqrt{tb^n}\right) \cdot \exp[(-q^2\sigma^2/2)m(\mathcal{L}(0, tb^n))]. \quad (96)$$

The only dependence on parameters enters through $m(\mathcal{L}(0, tb^n))$ and through ξ^2. Notably

$$\begin{aligned} b^n\xi^2 &= q^2\sigma^2 tb^n m(\mathcal{L}(0,1)) \\ &= q^2\sigma^2 b^n \int_{b^{n+1}}^{b^n} g(r)dr \\ &= q^2\sigma^2 \int_b^1 g^{(n)}(r)dr. \end{aligned}$$

□

APPENDIX IV
EXAMPLE OF A DIRECT COMPUTATION IN A NON-SCALE-INVARIANT CASE

We give below a direct derivation of the scaling of $\mathbb{E}A(t)^2$ for Example 7 of Section III-E, the special non-scale-invariant case when $dm(t,r) = cdtdr/r^{2+\beta}$ for $-1 < \beta < 0$. A convergence criterion is given as well.

Using definition (27) of $A_r(t)$, we have

$$\begin{aligned}\mathbb{E}A_r(t)^2 &= \int_0^t\int_0^t \mathbb{E}[Q_r(u)Q_r(v)]dudv \\ &= \int_0^t\int_0^t \exp\{-\varphi(2)m(\mathcal{C}_r(u)\cap\mathcal{C}_r(v))\}dudv. \end{aligned} \quad (97)$$

As a first step toward the autocorrelation $\mathbb{E}[Q_r(u)Q_r(v)]$, we note that for u and v such that $|u - v| \le 1$

$$m(\mathcal{C}_r(u)\cap\mathcal{C}_r(v)) = \int_{\max(r,|u-v|)}^1 \frac{c}{s^{2+\beta}}(s - |u-v|)\,ds. \quad (98)$$

$m(\mathcal{C}_r(u)\cap\mathcal{C}_r(v))$ is simply zero for $|u - v| \ge 1$. This yields whenever $\beta \notin \{-1, 0\}$: see (99) at the bottom of the page. As a consequence, we get (100), at the bottom of the following page. Then, $\mathbb{E}A_r(t)^2$ in (97) decomposes in the sum of two integrals on disjoint domains $\mathcal{E}_1$ and $\mathcal{E}_2$

$$\mathcal{E}_1 = \{(u,v) \in [0,\ t]^2 : r \le |u - v| \le 1\} \quad (101)$$
$$\mathcal{E}_2 = \{(u,v) \in [0,\ t]^2 : |u - v| < r\}. \quad (102)$$

First, using the changes of variables

$$\begin{cases} w = u - v \\ z = \frac{u+v}{2} \end{cases}$$

$$m(\mathcal{C}_r(t)\cap\mathcal{C}_r(s)) = \begin{cases} c\left[\frac{1-|t-s|^{-\beta}}{-\beta} + \frac{|t-s|}{1+\beta}\left(1 - |t-s|^{-(1+\beta)}\right)\right], & \text{for } r \le |t-s| < 1 \quad \text{(a)} \\ c\left[\frac{1-r^{-\beta}}{-\beta} + \frac{|t-s|}{1+\beta}\left(1 - r^{-1-\beta}\right)\right], & \text{for } 0 \le |t-s| \le r. \quad \text{(b)} \end{cases} \quad (99)$$

the integral over $\mathcal{E}_2$ yields

$$\int\int_{\mathcal{E}_2} \mathbb{E}[Q_r(u)Q_r(v)]dudv = 2\exp\left(\frac{c\varphi(2)}{\beta}(1-r^{-\beta})\right) \times \int_0^r dw(t-w)\exp\left[-\frac{c\varphi(2)|u-v|}{1+\beta}(1-r^{-(1+\beta)})\right]. \quad (103)$$

which vanishes as r tends to zero whenever $\beta \le 0$ and diverges to infinity as r tends to zero whenever $\beta > 0$. Thus, the limit $r \to 0$ makes sense for $\beta \le 0$ only.

Second, the integral over $\mathcal{E}_1$ yields

$$\int\int_{\mathcal{E}_1} \mathbb{E}[Q_r(u)Q_r(v)]dudv = 2e^{c\varphi(2)/\beta} \times \int_r^t dw(t-w)\exp\left[-\frac{c\varphi(2)}{\beta(1+\beta)}w^{-\beta}\right] \cdot \exp\left(-\frac{c\varphi(2)}{1+\beta}w\right). \quad (104)$$

For $w \ll \left|\frac{1+\beta}{c\varphi(2)}\right|$, we can use the following approximation:

$$\exp\left(-\frac{c\varphi(2)}{1+\beta}w\right) = 1 + O(w) \quad (105)$$

where $O(w)$ can be bounded by κw for some constant κ. Thus,

$$\int\int_{\mathcal{E}_1} \mathbb{E}[Q_r(u)Q_r(v)]dudv \simeq 2e^{c\varphi(2)/\beta} \times \int_r^t dw\left(\sum_{n=0}^{\infty}\frac{(\gamma w^{-\beta})^n}{n!}t - \sum_{n=0}^{\infty}\frac{(\gamma w^{-\beta})^n}{n!}w\right) \quad (106)$$

where $\gamma = -\frac{c\varphi(2)}{\beta(1+\beta)}$. For $-1 < \beta < 0$

$$\int\int_{\mathcal{E}_1} \mathbb{E}[Q_r(u)Q_r(v)]dudv \simeq e^{c\varphi(2)/\beta}\sum_{n=0}^{\infty}\frac{\gamma^n}{n!}\frac{1}{(1-\beta n)\left(1-\frac{\beta}{2}n\right)}t^{2-\beta n} + O(r) \quad (107)$$

where $|O(r)| \le \kappa r$ for some constant κ. Only the first term will remain in the limit $r \to 0$. Let $\alpha = \frac{\varphi(2)}{\beta} \ge 0$ and $-1 < \beta < 0$ and[11]

$$\begin{cases} f(t) = \exp\left(-\alpha t^{-\beta}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{\alpha^n}{n!}t^{-\beta n} \\ g(t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(1+\beta)^n(1-\beta n)\left(1\frac{\beta}{2}n\right)}\frac{\alpha^n}{n!}t^{-\beta n}. \end{cases} \quad (108)$$

[11]It is of interest to note that $g(t)$ can be easily obtained by numerical integration from the second derivative of $t^2g(t)$ given by

$$\frac{d^2}{dt^2}[t^2g(t)] = 2\exp\left[-\frac{\alpha}{1+\beta}t^{-\beta}\right].$$

To show that the scaling described by (40) of Theorem 1 is valid for $q = 2$ is now equivalent to show that $|f(t)-g(t)|/f(t) \ll 1$ for $t \le 1$. This is done by studying the sum

$$S(t) = f(t) - g(t) = \sum_{n=0}^{\infty}(-1)^n\frac{\alpha^n}{n!}\left(1-\frac{1}{C(n)}\right)t^{-\beta n} \quad (109)$$

where

$$C(x) = (1+\beta)^x(1-\beta x)\left(1-\frac{\beta}{2}x\right). \quad (110)$$

The study of $C(x)$ for $x \ge 0$ shows that for any $\beta < 0$ there exists some $x_o \ge 0$ such that $\forall\, x \ge x_o$, $C(x) < 1$ so that $(1-1/C(x)) < 0$. Then using usual criteria of convergence for partial sums of alternate series for $n \ge x_o$, one shows that

$$\frac{|S(t)|}{f(t)} \underset{t\to 0}{\sim} \left|\alpha\left(1-\frac{1}{C(1)}\right)\right|t^{-\beta}. \quad (111)$$

Note that $(1-1/C(1))$ remains small in general since it is less than 0.15 for $-0.83 < \beta < 0$. This result can be made even more precise for a chosen example. For instance, when $\beta = -0.4$, $\varphi(2) = -0.2$ and $c = 1$ as in Example 7 of Sections III-E and -V, one obtains that for any $t \le 1$

$$|S(t)| \le \alpha\left(1-\frac{1}{C(1)}\right)t^{-\beta} - \frac{\alpha^2}{2}\left(1-\frac{1}{C(2)}\right)t^{-2\beta} \quad (112)$$

so that

$$\frac{|S(t)|}{f(t)} \le 0.03, \qquad \forall\, t \le 1. \quad (113)$$

Moreover, the approximation (105) is true for $t \ll \left|\frac{1+\beta}{\varphi(2)}\right| = 3$ that is, for $t \lesssim 0.3$. Thus, we can consider with well-controlled accuracy that $f(t) \simeq g(t)$ (see Fig. 10). Thus, we can consider with well-controlled accuracy that $f(t) \simeq g(t)$. Finally, using (97), (103), (107), and (111) for $-1 < \beta < 0$

$$\mathbb{E}[A(t)^2] \simeq e^{c\varphi(2)/\beta}t^2\exp\left(\frac{1-t^{-\beta}}{-\beta}\right) \quad (114)$$

for $t \ll \left|\frac{1+\beta}{c\varphi(2)}\right|$. This exactly corresponds to the scaling behavior described in (40) of Theorem 1. Thus, we have obtained by direct computation the non-scale-invariant behavior observed on $\mathbb{E}A(t)^2$ in Example 7 of Section III-E.

Appendix V
Proof of Proposition 3 ($H = 1/2$)

Let us consider $V_{1/2,r}(t) = B(A_r(t))$ and $Z_r(t)$.

Conditioning on knowing A_r, note that $\{V_{1/2,r}(t)|\mathcal{F}_r\}$, where $\mathcal{F}_r$ denotes the natural filtration, is a zero-mean

$$\mathbb{E}[Q_r(u)Q_r(v)] = \begin{cases} e^{c\varphi(2)/\beta}\exp\left(-\frac{c\varphi(2)|u-v|}{1+\beta}\right)\exp\left[-\frac{c\varphi(2)|u-v|^{-\beta}}{\beta(1+\beta)}\right], & \text{for } r \le |u-v| \le 1 \\ \exp\left(\frac{c\varphi(2)}{\beta}(1-r^{-\beta})\right)\exp\left[-\frac{c\varphi(2)|u-v|}{1+\beta}(1-r^{-(1+\beta)})\right], & \text{for } |u-v| \le r. \end{cases} \quad (100)$$

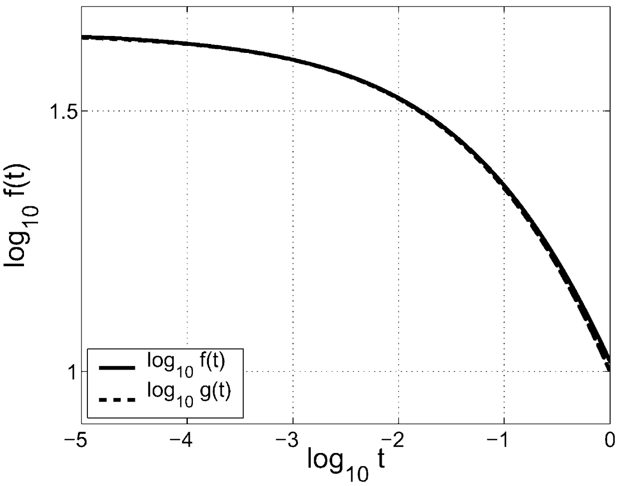

Fig. 10. $f(t)$ compared to $g(t)$ for the choice $\beta = -0.4$, $c = 1$, $\varphi(2) = -0.2$. This figure may be compared to Fig. 4 (c).

Gaussian process. Using $\mathbb{E}B(t)B(s) = \sigma^2 \min(t,s)$ with $\sigma^2 = \mathrm{var}(B(1)) = \mathbb{E}[|B(1)|^2]$ we find

$$\mathbb{E}[B(A_r(s))B(A_r(t))\big|\mathcal{F}_r] = \sigma^2 \min(A_r(t), A_r(s)). \quad (115)$$

Together with $\mathbb{E}[A_r(t)] = t$ we get

$$\mathbb{E}B(A_r(t))B(A_r(s)) = \sigma^2 \min(t,s).$$

Let us now turn to Z_r by considering $\{Z_r(t)\big|\mathcal{F}_r\}$. The integrand of the Ito integral in (60) being now deterministic this is a zero-mean Gaussian process. For simplicity, assume $s < t$ for the moment. We use a well-known rule of the Ito integral

$$\begin{aligned}\mathbb{E}&\left[\int_0^t f(u)dB(u)\int_0^s g(v)dB(v)\right]\\ &= \mathbb{E}\left[\int_0^s f(u)dB(u)\int_0^s g(v)dB(v)\right]\\ &\quad + \mathbb{E}\left[\int_s^t f(u)dB(u)\int_0^s g(v)dB(v)\right]\\ &= \sigma^2\int_0^s f(u)g(u)du. \end{aligned} \quad (116)$$

In the second step we used that the integrals over disjoint intervals are independent and zero mean. From this we obtain

$$\begin{aligned}\mathbb{E}[Z_r(t)Z_r(s)\big|\mathcal{F}_r] &= \sigma^2\int_0^s \sqrt{Q_r(u)}\sqrt{Q_r(u)}du\\ &= \sigma^2 A_r(s) = \sigma^2\min(A_r(t), A_r(s))\end{aligned} \quad (117)$$

that coincides with (115).

Conditioned on knowing A_r, both processes $\{V_{1/2,r}(t)|\mathcal{F}_r\}$ and $\{Z_r(t)|\mathcal{F}_r\}$ are Gaussian with identical autocorrelation $\sigma^2\min(A_r(t), A_r(s))$. They are, thus, identical in the sense of finite-dimensional distribution, and so must be the unconditional processes $V_{1/2,r}(t) = B(A_r(t))$ and $Z_r(t)$. Furthermore, $\mathbb{E}V_{1/2,r}(t)V_{1/2,r}(s) = \sigma^2\min(s,t)$.

In the standard Brownian case, $H = 1/2$, we point out that the increments $V_{1/2,r}(t) - V_{1/2,r}(s)$ and $V_{1/2,r}(v) - V_{1/2,r}(u)$ are (second-order) *uncorrelated* whenever $u < v \leq s < t$; this follows easily by conditioning on $\mathcal{F}_r$ using the independence of the increments of the ordinary Brownian motion B. However, they are *not independent* and inherit higher order correlations from $Q_r(s)$. Mandelbrot calls this the "blind spot of spectral analysis" (see also [1]).

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References

[1] R. H. Riedi, "Multifractal processes," in *Theory and Applications of Long Range Dependence*, P. Doukhan, G. Oppenheim, and M. S. Taqqu, Eds. Boston, MA: Birkhauser, 2003, pp. 625–716.

[2] B. B. Mandelbrot and J. W. V. Ness, "Fractional brownian motion, fractional noises and applications," *SIAM Rev.*, vol. 10, pp. 422–437, 1968.

[3] D. Veitch, P. Abry, P. Flandrin, and P. Chainais, "Infinitely divisible cascade analysis of network traffic data," in *Proc. IEEE Int. Conf. Acoustics, Speech, and Signal Processing—ICASSP*, Istanbul, Turkey, 2000.

[4] P. Abry, R. Baraniuk, P. Flandrin, R. Riedi, and D. Veitch, "The multiscale nature of network traffic: Discovery, analysis, and modeling," *IEEE Signal Process. Mag.*, vol. 19, no. 3, pp. 28–46, May 2002.

[5] B. Castaing, Y. Gagne, and E. Hopfinger, "Velocity probability density functions of high reynolds number turbulence," *Physica D*, vol. 46, pp. 177–200, 1990.

[6] B. Castaing, "The temperature of turbulent flows," *J. Phys. II France*, vol. 6, pp. 105–114, 1996.

[7] E. A. Novikov, "Infinitely divisible distributions in turbulence," *Phys. Rev. E*, vol. 50, no. 5, pp. R3303–R3305, 1994.

[8] A. Arneodo, J. Muzy, and S. Roux, "Experimental analysis of self-similarity and random cascade processes: Application to fully developed turbulence data," *J. Phys. II France*, vol. 7, pp. 363–370, 1997.

[9] P. Chainais, "Cascades log-infiniment divisibles et analyse multirésolution. Application à l'étude des intermittences en turbulence," Ph.D. dissertation, E.N.S, Lyon, France, 2001.

[10] J. Muzy, E. Bacry, and A. Arneodo, "The multifractal formalism revisited with wavelets," *Int. J. Bifurc. and Chaos*, vol. 4, no. 2, pp. 245–301, 1994.

[11] S. Jaffard and Y. Meyer, "Wavelet methods for pointwise regularity and local oscillations of functions," *Mem. Amer. Math. Soc.*, vol. 123, no. 587, Sep. 1996.

[12] P. Chainais, P. Abry, and J. Pinton, "Intermittency and coherent structures in a turbulent flow: A wavelet analysis of joint pressure and velocity measurements," *Phys. Fluids*, vol. 11, no. 11, pp. 3524–3539, 1999.

[13] P. Abry, P. Flandrin, M. Taqqu, and D. Veitch, "Wavelets for the analysis, estimation and synthesis of scaling data," in *Self-Similar Network Traffic and Performance Evaluation*, K. Park and W. Willinger, Eds. New York: Wiley, 2000.

[14] B. B. Mandelbrot, "Intermittent turbulence in self similar cascades: Divergence of high moments and dimension of the carrier," *J. Fluid. Mech.*, vol. 62, pp. 331–331, 1974.

[15] ——, "Iterated random multiplications and invariance under randomly weighted averaging," *Compt. Rend. (Paris)*, vol. 278A, pp. 289–292, 1974.

[16] J.-P. Kahane and J. Peyrière, "Sur certaines martingales de Benoit Mandelbrot," *Adv. Math.*, vol. 22, pp. 131–145, 1976.

[17] F. Schmitt and D. Marsan, "Stochastic equations generating continuous multiplicative cascades," *Eur. Phys. J. B*, vol. 20, pp. 3–6, 2001.

[18] J. Barral and B. Mandelbrot, “Multiplicative products of cylindrical pulses,” *Probab. Theory Rel. Fields*, vol. 124, pp. 409–430, 2002.

[19] E. Bacry, J. Delour, and J. Muzy, “Multifractal random walk,” *Phys. Rev. E*, vol. 64, p. 026103, 2001.

[20] J. Delour, “Processus aléatoires auto-similaires: Applications en turbulence et en finance,” Ph.D. dissertation, Univ. Bordeaux I, Bordeaux, France, 2001.

[21] J. Muzy and E. Bacry, “Multifractal stationary random measures and multifractal random walks with log-infinitely divisible scaling laws,” *Phys. Rev. E*, vol. 66, 2002.

[22] P. Chainais, R. Riedi, and P. Abry, “Scale invariant infinitely divisible cascades,” in *Proc. Int. Symp. Physics in Signal and Image Processing*, Grenoble, France, Jan. 2003.

[23] E. Bacry and J. Muzy, “Log-infinitely divisible multifractal processes,” *Commun. Math. Phys.*, vol. 236, pp. 449–475, 2003.

[24] J. Barral and B. Mandelbrot, “Multiplicative Products of Cylindrical Pulses,” Cowles Foundation, discussion paper 1287, 1999.

[25] P. Chainais, R. Riedi, and P. Abry, “Compound Poisson cascades,” in *Colloque Autosimilarité et Applications*, France, 2002.

[26] J. Barral, “Poissonian products of random weights: Uniform convergence and related measures,” *Rev. Mat. Iberoamericana*, vol. 19, pp. 1–44, 2003.

[27] P. Chainais, R. Riedi, and P. Abry, “Warped infinitely divisible cascades: Beyond scale invariance,” *Traitement du Signal*, vol. 22, no. 1, 2004.

[28] J.-P. Kahane, “Positives martingales and random measures,” *Chinese Ann. Math.*, vol. 8B, pp. 1–12, 1987.

[29] R. Riedi, M. S. Crouse, V. Ribeiro, and R. G. Baraniuk, “A multifractal wavelet model with application to TCP network traffic,” *IEEE Trans. Inf. Theory (Special Issue on Multiscale Statistical Signal Analysis and Its Applications)*, vol. 45, no. 3, pp. 992–1018, Apr. 1999.

[30] A. Fan, “Sur les chaos de Lévy stables d’indice $0 < \alpha < 1$,” *Ann. Sci. Math. Québec*, vol. 21, pp. 53–66, 1997.

[31] J. Kahane, “Sur le chaos multiplicatif,” *Ann. Sci. Math. Québec*, vol. 9, pp. 105–150, 1985.

[32] W. Feller, *An Introduction to Probability Theory and Its Applications*. New York, London, Sidney: Wiley, 1966, vol. 2.

[33] A. N. Kolmogorov, “A refinement hypothesis concerning the local structure of turbulence in a viscous incompressible fluid at high reynolds number,” *J. Fluid Mech.*, vol. 13, pp. 82–85, 1962.

[34] G. Samorodnitsky and M. Taqqu, *Stable Non-Gaussian Random Processes*. New York: Chapman and Hall, 1994.

[35] D. Schertzer and S. Lovejoy, “Physical modeling and analysis of rain and clouds by anisotropic scaling multiplicative processes,” *J. Geophys. Res.*, vol. 92, pp. 9693–9693, 1987.

[36] Y. Saito, “Log-gamma distribution model of intermittency in turbulence,” *J. Phys. Soc. Japan*, vol. 61, no. 2, pp. 403–406, 1992.

[37] Z. S. She and E. C. Waymire, “Quantized energy cascade and log-Poisson statistics in fully developed turbulence,” *Phys. Rev. Lett.*, vol. 74, no. 2, pp. 262–265, 1995.

[38] Z. S. She and E. Levêque, “Universal scaling laws in fully developed turbulence,” *Phys. Rev. Lett.*, vol. 72, no. 3, pp. 336–339, 1994.

[39] B. Dubrulle, “Intermittency in fully developed turbulence: Log-Poisson statistics and scale invariance,” *Phys. Rev. Lett.*, vol. 73, pp. 969–969, 1994.

[40] U. Frisch, *Turbulence. The Legacy of A. Kolmogorov*. Cambridge, U.K.: Cambridge Univ. Press, 1995.

[41] L. C. G. Rogers and D. Williams, *Diffusions, Markov Processes and Martingales*, 2nd ed. New York: Wiley, 1994, vol. 1, Foundations.

[42] B. B. Mandelbrot, “A multifractal walk down Wall Street,” *Scientific Amer.*, vol. 280, no. 2, pp. 70–73, Feb. 1999.

[43] A. Wood and G. Chan, “Simulation of stationary Gaussian processes in $[0, 1]^d$,” *J. Comput. Graph. Statist.*, vol. 3, pp. 409–432, 1994.

[44] G. Chan and A. Wood, “An algorithm for simulating stationary gaussian random fields,” *Appl. Statist.*, vol. 46, pp. 171–181, 1997.

[45] B. Lashermes, P. Abry, and P. Chainais, “New insight in the estimation of scaling exponents,” *Int. J. Wavelets, Multires. Inf. Process.*, vol. 2, no. 4, pp. 497–523, Dec. 2004.

[46] B. Rajput and J. Rosinski, “Spectral representations of infinitely divisible processes,” *Probab. Theory Related Fields*, vol. 82, pp. 451–487, 1989.

[47] P. Mannersalo, R. Riedi, and I. Norros, “Multifractal products of stochastic processes: Construction and some basic properties,” *Appl. Probab.*, vol. 34, no. 4, pp. 1–16, 2002.

[48] M. Leadbetter, *Extremes and Related Properties of Random Sequences and Processes*. New York: Springer-Verlag, 1982, Springer Series in Statistics.