Infinitely Divisible Shot-Noise: Modeling Fluctuations in Networking and Turbulence

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Abstract. This paper provides an introduction to recent advances in the study of scale-invariance and related phenomena, namely the concept of infinitely divisible scaling which encompasses statistical self-similarity and multifractal scaling. Further the paper develops path properties and scaling of Poisson products of multiplicative and exponential shot-noise processes.

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1. INFINITELY DIVISIBLE SCALING: A VERSATILE CONCEPT

Scaling and related phenomena form a striking and crucial component of empirical data observed in a wide variety of applications ranging from natural phenomena such as hydrodynamic turbulence [1], to purely human phenomena created by mankind's activities such as computer networks [2, 3, 4] and financial markets [5, 6]. Often, scaling directly impacts performance, e.g., leading to high volatility in markets and to large waiting queues in networking. Versatile tools of analysis and models are, therefore, imperative towards improving our understanding and physical interpretations.

Scaling. As of today, the concept of *infinitely divisible scaling* of a process $\{X(t)\}_t$ appears to be the most general of its kind, introduced at the end of the out-going millennium [8, 7]:

$$\mathbb{E}|A(t+\delta)-A(t)|^q = c_q \exp[-H(q)\cdot n(\delta)]. \tag{1}$$

This framework encompasses at the same time statistical self-similarity as exhibited, e.g., by fractional Brownian motion where $H(q) = H \cdot q$ as well as multifractal scaling as exhibited, e.g., by the Martingale of Mandelbrot [9, 10, 11] (see below).

Note that for both, self-similar processes and multifractals, the scaling is in terms of a power-law since $n(\delta) = -\log(\delta)$. The extra degree of freedom of infinitely divisible scaling in terms of $n(\delta)$ was found highly useful for the analysis and modelling of empirical data in turbulence and computer network traffic [8, 7, 12, 4, 13].

Cascades. The basic structure of a *cascade* $\{Q_r(t)\}_t$ consists of multiplying building blocks, i.e. stochastic processes $P_i(t)$, as follows:

$$Q_r(t) := \prod_{R_i>r} P_i(t) := \prod_{R_i>r} \pi\left(W_i, \tfrac{t-T_i}{R_i}\right). \tag{2}$$

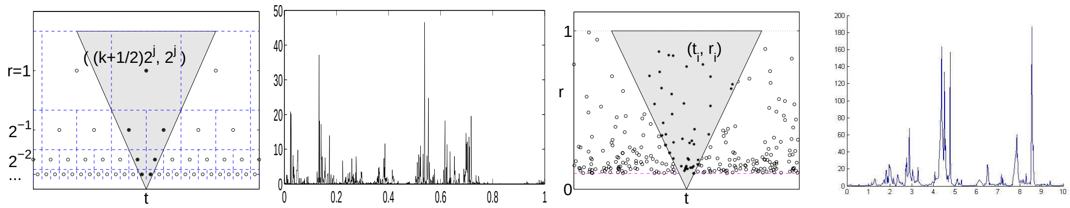

FIGURE 1. From left: Dyadic geometry of the Martingale of Mandelbrot, and a realization; scale-invariant Poisson random geometry of a shot-noise cascade, and a realization (far right) with multiplicative Gaussian kernel ($c = .5$). [plots of geometries courtesy P. Chainais [13]]

Here, the so-called *pulse* $\pi(w,\cdot)$ is a non-negative function converging to 1 at infinity, with parameter w. In the building block P_i, the parameter W_i plays the role of an amplitude, i.e., $W_i = \max_t P_i(t)$. Clearly, T_i is a location parameter while R_i represents the time scale over which P_i is significantly different from 1. To gain some intuition, one may interpret $Q_r(t)$ as the intensity resulting from scattering light at identical shapes, placed at random locations T_i and at random heights R_i, with random transparency W_i (see [14] and references therein for further physical interpretations and applications).

The most basic pulse, called *cylindrical pulse* by Mandelbrot [10] leads to a simple form of Q_r:

$$\pi(w,s) = \begin{cases} w & \text{for } 0 \le s < 1 \\ 1 & \text{otherwise.} \end{cases} \qquad\qquad Q_r(t) = \prod_{R_i > r;\ T_i < t < T_i + R_i} W_i \tag{3}$$

The well-known *Martingale of Mandelbrot* [9] is recovered by choosing W_i to be independent identically distributed (i.i.d.) random variables with mean 1, $R_i = 1/2^n$ for $2^n - 1 \le i < 2^{n+1} - 1$ and $T_{2^n+k-1} = k/2^n$. In this simple deterministic case, Q_r can be interpreted as a density that is uniform over each dyadic interval $[k/2^n, (k+1)/2^n[$. Since $\mathbb{E}[A(1/2^n)^q] = \mathbb{E}[W^q]^n$ for $A(t) = \lim_{r\to 0}\int_0^t Q_r(s)ds$, one finds (1) with $H(q) = -\log\mathbb{E}[W^q]$ and $n(\delta) = -\log(\delta)$, at least for $\delta = 1/2^n$.

Two inherent properties of the martingale of Mandelbrot have limited its applicability to real world problems, namely the rigid, dyadic geometry restricting exact scaling to binary scales and imposing non-stationarity (see Figure 1 left), as well as the scaling in form of a power-law.

A crucial step towards modeling infinitely-divisible scaling over all scales and beyond power-laws was taken with the introduction of *Compound-Poisson-Cascades* (CPC) and their subsequential generalization to Infinitely-Divisible-Cascades [15, 16, 17, 10, 13].

The main novelty of CPC over the Martingale of Mandelbrot consists in choosing the location and scale parameters of the pulses according to a Poisson Point Process (see Figure 1 right). Notably, this implies that the cascade Q_r is made of a random number of factors. Most conveniently, one may think of the locations $\{T_i\}_i$ as forming a homogeneous Poisson Process of arrival rate λ_r on the real line and the scales R_i being i.i.d. marks with density $g_r(\cdot)$ on the interval $[r,1]$. Equivalently, the points $(T_i,R_i)_i$ form a Poisson point process, meaning that the number $N(E)$ of such points falling into a measurable set E in the plane is a Poisson random variable which is independent of

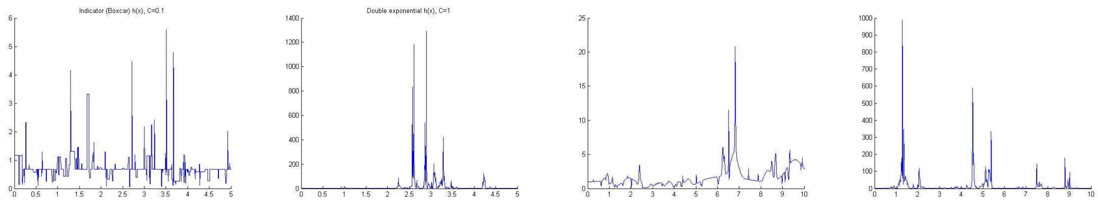

FIGURE 2. Effects of pulse and density: Realizations of scale-invariant (5) multiplicative shot-noise cascades, from left with cylindrical ($c = .1$), double exponential ($c = 1$), Gaussian ($c = .1$), and again Gaussian ($c = 1$) kernels, respectively. They are all normalized to mean 1.

$N(E')$ whenever E' and E are disjoint, and with mean

$$\mathbb{E}[N(E)] = m(E) := \int_E dm(t,a) := \int_E \lambda_r g_r(a) da dt, \tag{4}$$

Thereby, it is assumed that the multipliers $\{W_i\}_i$ are i.i.d. and independent of the Point Process $(T_i, R_i)_i$.

Today, both the scaling behavior of CPCs as well as their mathematical properties [16, 17, 13] are well understood. A special role plays the *scale-invariant case* where

$$dm(t,a) = \frac{c}{a^2} da dt \qquad \text{for } r < a < 1 \text{ and } t \in \mathbb{R}. \tag{5}$$

Since g_r forms a density on $[r,1]$ we must choose $\lambda_r = c(1/r - 1)$, meaning that for $r = 1/2^n$ there fall on average $c(2^n - 1)$ points T_i into a t-interval of unit length, in rough agreement with the deterministic geometry of the Martingale of Mandelbrot.

The importance of the scale invariant point density lies in its close connection to asymptotic power-laws. It has been studied most intensively [16, 6, 17, 13]. Notably, exact power-laws are given in [16]; the connection between scaling and local fractal behavior is rigorously established via the multifractal formalism in [17]. Also, scaling beyond power-laws with densities $dm(t,a)$ deviating from (5) are reported in [13].

Shot-noise cascades. More recently, attention has focussed on extending the CPC to cascades with pulses π different from the cylindrical pulse (3) [15, 17, 13, 14]. Thereby, two versions offer themselves, which we term the *multiplicative shot-noise* and *exponential shot-noise*, where

$$\pi(w,t) := \begin{cases} 1 + (w-1)h(t) & \text{multipicative shot-noise,} \\ w^{k(t)} & \text{exponential shot-noise.} \end{cases} \tag{6}$$

The functions h and k are called *kernel* and are assumed to be non-negative, bounded by 1 and to converge to zero at infinity. When choosing either kernel to be the indicator of the unit interval $1_{[0,1]}(\cdot)$, the pulse reduces to the basic cylindrical shape (3). Both choices lead to similar convergence properties. Notably, the exponential shot-noise cascade finds a more natural interpretation in the more general setting of infinitely divisible cascades [13, 14]. For examples, see Figure 2.

2. SHOT-NOISE CASCADES

For clarity we emphasize that we assume that Q_r is a multiplicative cascade of the form of (2) where the pulse π is a shot-noise with kernel h, respectively k, as in (6). We further assume that locations T_i and scales R_i form a Poisson point process as in (4) and that the amplitudes or weights W_i are i.i.d. and independent of the point process. Note that we do *not* assume $\mathbb{E}[W_i] = 1$, nor scale invariance, unless indicated.

Also, in contrast with much of the existing literature we do not assume that the pulse π be compactly supported. While such an assumption avoids technical subtleties it also precludes the use of simple and natural exponential, Gaussian and wavelet transform kernels.

Convergence and Moments. In general, for fixed t and r, $Q_r(t)$ given in (2) may consist of infinitely many factors. Alluding to martingale techniques we find that arguing for convergence goes hand in hand with the computation of moments. We recall a well-known fact saying that conditioning on knowing the number $N_{[a,b]}$ of points T_i that fall into the interval $[a,b]$, these points T_i become i.i.d. and distributed as a uniform variable U on $[a,b]$. Fixing q, r and t, we find:

$$\begin{aligned} c_q(a,b) &:= \mathbb{E} \prod_{a<T_i<b} P_i^q(t) = \mathbb{E}\mathbb{E}\left[\prod_{a<T_i<b} P_i^q(t) \Big| N_{[a,b]} \right] = \mathbb{E}\left[\left(\mathbb{E}[\pi(W, \tfrac{t-U}{R})^q] \right)^{N_{[a,b]}} \right] \\ &= \exp\left[\lambda_r (b-a)(\mathbb{E}[\pi(W, \tfrac{t-U}{R})^q] - 1) \right] = \exp\left[\lambda_r \mathbb{E}[\int_a^b \pi(W, \tfrac{t-u}{R})^q - 1 du] \right] \\ &= \exp\left[\lambda_r \mathbb{E}[R \cdot \int_{(t-b)/R}^{(t-a)/R} \pi(W,v)^q - 1 dv] \right] \end{aligned} \tag{7}$$

The independence properties of Poisson point processes imply that $c_q(a,c) = c_q(a,b) \cdot c_q(b,c)$ for $a < b < c$ and that the normalized products $\{\frac{1}{c_1(a,b)} \prod_{a<T_i<b} P_i(t)\}_{a<0<b}$ form a continuous, positive, L_1-bounded martingale with respect to its natural filtration. By Doob's martingale convergence theorem it must converge almost surely as $-a, b \to \infty$ to a limiting L_1 random variable. Note that the convergence is not necessarily in L_1; however, the limit being in L_1 it must be almost surely finite. Provided the normalization constants, i.e. $c_1(a,b)$, converge to a finite limit, the products $\prod_{a<T_i<b} P_i(t)$ must themselves converge almost surely. We conclude:

Theorem 1 *Fix t and r. Assume that $\mathbb{E}\int_{-\infty}^{\infty} |\pi(W,v) - 1| dv < \infty$. Then, the compound Poisson cascade specified by (2) and (4) is well defined as an almost sure limit. If also*

$$\mathbb{E}\int_{-\infty}^{\infty} |\pi(W,v)^q - 1| dv < \infty \tag{8}$$

for some $q > 0$, then the product $Q_r(t)$ in (2) is an L_p-limit for $0 < p < q$ and

$$\mathbb{E}[Q_r(t)^p] \quad = \quad \exp\left[\lambda_r \mathbb{E}[R] \cdot \mathbb{E}\int_{-\infty}^{\infty} \pi(W,v)^p - 1 dv \right] \tag{9}$$

Sufficient for (8) is $\mathbb{E}[W^n] < \infty$ for some integer $n \geq q$ and $\int h(u)du < \infty$ in the case of a multiplicative shot-noise cascade, respectively $M(s) = \mathbb{E}[e^{s|logW|}] < \infty$ for some $s > 0$ and $\int k(u)du < \infty$ in the case of an exponential shot-noise cascade.

Proof
Under the moment condition (8) dominated convergence implies that $c_q(a,b)$ converges as $-a, b \to \infty$. For $q = 1$ this implies almost sure convergence of the product $Q_r(t)$ as indicated above. In addition, it implies that the family of products $\prod_{a<T_i<b} P_i(t)^p$ is uniformly integrable; thus, the product $Q_r(t)$ converges in L_p. To verify the sufficient conditions for (8) one uses the binomial expansion in the multiplicative case, respectively the inequality $|e^a - 1| \leq e^{|a|} - 1$ for all $a \in \mathbb{R}$ in the exponential case, noting that $M(\cdot)$ is differentiable around 0. ♢

Note that the moments do not depend on t, in agreement with the translation invariance along the t-variable of the Poisson process (see (4)). Similarly one finds, provided $Q_r(t)$ is an L_2-limit, that

$$\mathbb{E}[Q_r(t)Q_r(s)] \quad = \quad \exp\left(\lambda_r \mathbb{E}[R \cdot \int_{-\infty}^{\infty} \pi(u)\pi(u + \tfrac{t-s}{R}) - 1 du]\right) \tag{10}$$

Path-properties and Scaling. The path oscillations of stochastic processes formed by cascades have attracted quite some attention. We start by establishing that the CPC $\{Q_r(t)\}_t$ defined through (2) and (4) exists as a stochastic process and exhibits continuous, or at least cadlag paths under mild conditions.

Proposition 2 *Fix $r > 0$. Assume that the pulse branches $\pi(W,t)$ and $\pi(W,-t)$ are both non-increasing and convex for $t > c$ almost surely (a.s.), for some c. Assume that the product (2) converges a.s. and in L_1 for any fixed t. Then, the product (2) converges in the Skorohod topology, hence, $\{Q_r(t)\}_t$ is a cadlag process. If the pulse $\pi(W,\cdot)$ is in addition continuous a.s., then $\{Q_r(t)\}_t$ possesses a.s. continuous paths.*

Sufficient conditions for the properties of the pulse branches in the shot-nose cases are that the kernel branches be non-increasing and convex for $t > c$.

Proof The main idea is to note that the largest variations of the $P_i(t)$ with $|T_i| > \tau$ (see (2)) over the interval $[-\tau, \tau]$ are at the boundaries, due to convexity. ♢

Let us now turn to the limit of $r \to 0$ and to scaling. To this end, let $\bar{Q}_r(t) := Q_r(t)/\mathbb{E}[Q_r(t)]$ be the normalized cascade. Being a positive continuous martingale indexed by $r > 0$, its distributional limit $A(t) = \lim_{r\to 0} \int_0^t \bar{Q}_r(s)ds$ exists almost surely. Tools towards assessing path regularity, such as the Kolmogorov regularity theorem or more generally, multifractal analysis and infinitely divisible scaling, rely on the knowledge of scaling behavior of the moments of $\mathbb{E}[|A(t+\delta) - A(t)|^q]$ as $\delta \to 0$. To this end, we follow [13, III.C] closely and find

Theorem 3 *Let $A(t)$ be as above, assuming the scale-invariant case (2), (4), (5), $q > 0$.*

- *(Moment condition) Assume (8) and that $A(t)$ converges almost surely and in L_1.*
- *(Variational condition) Assume that $\pi(t)$ and $\pi(-t)$ are $C^1(\mathbb{R}^+)$ and non-increasing and convex for $t > c$ almost surely, for some c.*

Let $f(\delta) \sim g(\delta)$ stand for $C < \frac{f(\delta)}{g(\delta)} < 1/C$ for all $0 < \delta < 1$, where $0 < C < \infty$. Then, we find infinitely divisible scaling, at least approximately, of the form:

$$\mathbb{E}[|A(t+\delta) - A(t)|^q] \sim \delta^q \mathbb{E}\left[\overline{Q}_\delta(0)^q\right] = \delta^{q-c(\rho(q)-q\rho(1))} \tag{11}$$

where $\rho(q) = \frac{1}{c}\log_{1/\delta}(\mathbb{E}[Q_\delta(0)^q]) = \mathbb{E}[\int_{-\infty}^{\infty} \pi(W,u)^q - 1du]$.

Proof The main idea is to follow the argumentation of [13, III.C] verbatim. Notably, matters simplify here considerably due to scale-invariance. Basically, the main task is to show that

$$\mathbb{E} \sup_{0 \le s \le t} |Q_r(s)^q - Q_r(0)^q| \le t \cdot C' \tag{12}$$

where C' is independent of t. This task, in turn, is simplified as compared to [13] since the regularity assumptions here allow the use of the mean value theorem. Similar to the Skorohod argument above, consider blocks $P_i(t)$ with $|T_i| < \tau$ and $\tau < |T_i| < T$ and use Fatou to let $T \to \infty$. For the computation of $\rho(q)$ note that (5) implies $\lambda_r \mathbb{E}[R] = c \log \frac{1}{r}$ and use (9)). ◇

Details on the proofs of this section can be found in [18].

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