On the Broadcast Capacity of Multihop Wireless Networks: Interplay of Power, Density and Interference

Alireza Keshavarz-Haddad[†] Rudolf Riedi[†‡]
Department of Electrical and Computer Engineering[†] and Department of Statistics[‡]
Rice University, 6100 Main Street, Houston, TX 77005, USA

Abstract— **In this paper we study the *broadcast capacity* of multihop wireless networks which we define as the maximum rate at which broadcast packets can be generated in the network such that all nodes receive the packets successfully within a given time. To asses the impact of topology and interference on the broadcast capacity we employ the *Physical Model* and *Generalized Physical Model* for the channel. Prior work was limited either by density constraints or by using the less realistic but manageable Protocol model [1], [2]. Under the Physical Model, we find that the broadcast capacity is within a constant factor of the channel capacity for a wide class of network topologies. Under the Generalized Physical Model, on the other hand, the network configuration is divided into three regimes depending on how the power is tuned in relation to network density and size and in which the broadcast capacity is asymptotically either zero, constant or unbounded. As we show, the broadcast capacity is limited by distant nodes in the first regime and by interference in the second regime. In the second regime, which covers a wide class of networks, the broadcast capacity is within a constant factor of the bandwidth.**

I. Introduction

There has been a growing interest to understand the fundamental capacity limits of wireless networks [3]–[8]. Results on network capacity are not only important from a theoretical point of view but also provide guidelines for protocol design in wireless networks. Hitherto, most research on network capacity has focused on the capacity of unicast connections between random source and destination nodes.

In this paper we study the *broadcast capacity* $\lambda_{\mathcal{B}}(\mathbf{g})$ of multihop wireless networks, which we define as the maximum rate of generation of broadcast packets by a set of nodes $\mathcal{B}$ in the network such that all nodes receive the packets successfully. In this work, we consider two channel models which are known in the literature as the *Physical* and the *Generalized Physical Model*. For comparison purposes we will also refer to the simpler *Protocol Model*.

The Protocol Model incorporates interference through simple distances, allowing to apply the usual graph theoretical approaches. The Physical Model models interference more accurately, but still assigns a constant transmission rate once successful transmission is guaranteed. In this latter sense it is close to the Protocol Model. The Generalized Physical Model finally allows for a transmission rate that depends on the level of interference and the distance between sender and receiver and thus allows for a more precise assessment of the broadcast capacity.

To the best of our knowledge, only two papers so far study the capacity of wireless networks for broadcasting. The first paper [1] models the locations of the nodes via a Poisson point process and the channel via the Generalized Physical Model. The author of [1] studies the broadcast capacity $\lambda_{\mathcal{B}}(\mathbf{g})$ for the case when $\mathcal{B}$ consists of one single generating node, providing an upper bound for $\lambda_{\mathcal{B}}(\mathbf{g})$ and showing that this bound in achievable up to a constant factor under the assumption that the Poisson intensity of the nodes is fixed.

The second related paper [2] computes the broadcast capacity for an arbitrary connected network but assuming the somewhat simpler and less realistic *Protocol Model* for the channel. It shows that the broadcast capacity is within a constant factor of the wireless channel capacity and that it does not depend on radio range, number of nodes, and area of the network.

As the first contribution of this paper, we study the broadcast capacity under the Physical Model. This channel model has been used in earlier work such as [3], [9] but not for the study of broadcast capacity. We develop bounds of the broadcast capacity $\lambda_{\mathcal{B}}(\mathbf{g})$ for arbitrary connected wireless networks and for an arbitrary set of generating nodes $\mathcal{B}$ and arbitrary weight vector $\mathbf{g}$. Surprisingly, we find that the broadcast capacity does not change more than a constant factor when we vary the number of nodes, the transmission power and the area of the network provided we keep a level of connectivity made precise in the paper. Thus, we are able to confirm that the earlier similar results of [2] found for the simpler Protocol Model hold also under the more realistic Physical Models. For the special case of homogeneous networks with large number of nodes, we find that the broadcast capacity is $\Theta(W)$ where W denotes the wireless channel capacity in bits per second. Here, we adopt the standard notation from complexity theory where $O(.)$, $\Omega(.)$, and $\Theta(.)$ stand for asymptotic upper, lower, and tight bounds, respectively.

As our second contribution, we study the broadcast capacity under the Generalized Physical Model which is widely used in network capacity papers [1], [5], [10]. In this model the transmission rate is determined by the Shannon capacity formula and thus depends on the receiving power and the interference of other signals. As a result, *poor connectivity* of some nodes and *high interference* from simultaneous transmissions can limit the broadcast capacity considerably, which we will quantify explicitly. Due to the strong impact of the geometry of node arrangement in this model, one needs to make some assumption about the topology. In this paper we focus on homogeneous networks which is a popular model in network capacity papers.

We find under these assumptions *three power regimes* for the broadcast capacity in homogeneous networks with large number of nodes. In the first regime, low transmission power causes the isolation of some nodes in the sense of a limiting receiving rate from the rest of the nodes. These nodes then become bottlenecks for the broadcast capacity which is asymptotically zero. In the second regime, power is sufficiently elevated and tuned to the size and density of the network to provide every node in the network with a data reception rate within a constant factor of the bandwidth. At the same time, the network size imposes simultaneous transmissions in order to achieve a constant rate to all nodes. This regime corresponds to the most common settings in wireless networking and leads to a broadcast capacity within a constant factor of the bandwidth. Here, interferences of simultaneous transmissions is the limiting factor for the broadcast capacity. In the third regime, the transmission power is so significant in relation to the network size that all nodes receive a direct transmission from the broadcast source with high rate. We will show that in this regime a multi-hop broadcast does not increase the throughput by more than a constant over direct single transmission from the source.

Notably, no prior work identifies nor studies these three power regimes, to the best of our knowledge. More concretely, this paper applies to a broader context than [1] by not imposing restrictions on the asymptotic densities and by treating also the Physical Model. Indeed, though not demonstrated due to space constraints, the low power regime mentioned above can be established also for a network modeled by a Poisson point process, thus including the results of [1] as a special case. The present work also goes beyond [2] which was based on the Protocol Model by assuming the more realistic Physical and Generalized Physical Models for wireless channels.

The paper is organized as follows. In Section II we summarize existing work on the network capacity. We introduce a network model and define relevant terms in Section III. In Section IV we compute upper and lower bounds for broadcast capacity which apply to arbitrary wireless networks under the Physical Model. Section V computes the broadcast capacity of homogeneous networks under the Generalized Physical Model as the number of nodes grows. Finally, we conclude in Section VI. All proofs are placed in the Appendix.

II. Related Work

Gupta and Kumar [3] studied the per node capacity for unicast connections between random sources and destinations in a static wireless network consisting of n nodes distributed in a fixed area. For planar networks this per node capacity decreases at least as $O(1/\sqrt{n})$ in an arbitrary network and even as $O(1/\sqrt{n\log(n)})$ in a random network as the number of nodes grows [3]; extensions to three dimensional space can be found in [9]. Both studies are based on the Protocol Model and the Physical Model. Later work [10] established that the same bounds still hold under the more accurate Generalized Physical Model. Also, [11] showed how to achievable asymptotically a throughput within a constant factor of these bounds in random networks. In another approach, [5] showed that per node capacity $\Omega(1/\sqrt{n})$ ia achievable using percolation theory techniques.

For mobile wireless networks, Grossglauser and Tse [4] proposed a mobility-based routing method in which the number of retransmissions for unicast communication between source and destination is reduced to 2. Thus, mobility can increase the per node capacity to $\Theta(1)$ provided that the packet delay is allowed to be arbitrarily large. Subsequent work analyzed the capacity under constraints on the delay or the mobility of the nodes [12]–[17].

Introducing a new direction in network capacity research, [1] studies the "broadcast capacity" of static wireless networks. The paper computes an upper bound for broadcast capacity using the Generalized Physical

Model. The paper proves that the broadcast capacity is asymptotically within a constant factor of this upper bound as the number of nodes goes to infinity provided that the Poisson intensity of the nodes remains fixed. However, the paper [1] does not address the case where the node intensity is allowed to tend to infinity as the number of nodes grows. In particular, it leaves open the question whether the upper bound $O(\log n)$ is achievable when the intensity is n. Further, and equally important, it neglects the effects of interference when computing the upper bound.

The present work goes considerably beyond the work of [1], taking into account interference and allowing for more general asymptotic node intensities. As we will see, there are three asymptotic regimes defined through power levels one of which covers [1], one of which resolves the open question left for future study in [1] and a third one that establishes a novel performance regime.

A related study [2] computes the broadcast capacity for an arbitrary network under the Protocol Model for the channel. The paper proves that the broadcast capacity lies within a constant factor of the wireless channel capacity (W) and that it does not depend on the radio range, the number of nodes in the network and the area of the network. Only the interference parameter has an effect on the broadcast capacity.

The present paper extends the prior work [2] by using the more realistic Physical and the Generalized Physical Model for the channel. Interestingly, we are able to show that bounds similar to [2] hold under these more realistic physical models. Moreover, under the Generalized Physical Model and under certain conditions on transmission power and network size, the broadcast capacity comes within a constant factor of the bandwidth. Due to space constraints we are unable to continue the study of the broadcast capacity of mobile networks initiated in [2] and leave it for future work.

Also, there is another paper on broadcast capacity [18] which studies a simplified channel and network model. The paper shows that per node broadcast capacity is $O(1/n)$ for such a model.

Note that all the above mentioned papers as well as this paper assume only point-to-point coding at the receivers. If the nodes are allowed to cooperate and use sophisticated multi-user coding then a per node capacity of a higher order than described above can be achieved [19]–[21]. A full discussion of these results is beyond the scope of this paper.

III. WIRELESS CHANNEL MODEL AND BASIC NOTIONS

In this section we describe the wireless network model and define several terms relevant to our analysis of broadcast capacity. We consider a wireless network consisting of n wireless nodes. Let X_i for $i = 1, 2, \ldots, n$ denote the location of the different nodes. For simplicity we also use X_i to refer to the i^{th} node itself. Note that in this paper we do all analysis in d-dimensional space, so $X_i \in \mathbb{R}^d$.

A. Network Graph

The following notions lie at the basis of many studies and will become useful here for comparison and computation purposes. We denote by $G(R)$ the geometric graph induced by distance where the vertices of $G(R)$ are the nodes of the network and where two distinct nodes X_i and X_j are *adjacent* in $G(R)$ if and only if $|X_i - X_j| \leq R$. Note that increasing R can only add edges to the ones of $G(R)$.

A *Minimum Connected Dominating Set* (MCDS) of a connected graph is a connected subset of nodes with the minimum size and the following property: every node of the graph is adjacent to at least one node in the subset. MCDS of a network graph uses minimum number of nodes to disseminate a broadcast packet to all nodes.

For clarity, we include the distance parameter R in the notation for the above defined sets. For example MCDS($R_{\max}$) is an MCDS of $G(R_{\max})$. We use the symbol $\#$ to denote the size of a set.

B. Homogeneous Networks

We now recall a wireless network model, called *homogeneous network*, which is widely employed in the study of network capacity. In this model, the number of nodes grows to infinity while the nodes are distributed uniformly in a d-dimensional sphere or cube of volume V_n. We point out that the volume V_n is allowed to vary with the number of nodes n. This model is closely related to modeling the node locations via a homogeneous Poisson point process with intensity $\frac{n}{V_n}$ per unit volume. Conditioned on knowing that there are exactly $\hat{n}$ points produced by the process, these points are independently and uniformly distributed in V_n. Note that $\hat{n}$ is a Poisson random variable with mean n.

For convenience, we introduce D as the diameter of V_n. For a cube

$$D = \sqrt{d} V_n^{1/d} \tag{1}$$

C. Channel Model

Network capacity papers usually model the wireless channel in three ways as we explained in the introduction. Here, we describe each model in details. Let $\mathcal{T}$ be the subset of nodes simultaneously transmitting at some given time instant. For convenience we call the case, where $\mathcal{T}$ contains only the sender, an *exclusive transmission*. We call a network *interference-free connected* whenever it is connected under the assumption of all transmissions being exclusive.

1) Protocol Model: A transmission from node $X_i \in \mathcal{T}$ to node X_j is modeled as successful if $|X_i - X_j| \leq R$ and if $|X_k - X_j| \geq (1+\Delta)R$ for all $X_k \in \mathcal{T}\backslash\{X_i\}$ where Δ is the *interference parameter*. In this model, the rate of a successful transmission is considered to be constant, denoted by W.

We also define R_c as the minimal R such that the network becomes interference-free connected under the Protocol Model *with high probability* (w.h.p.) as number of nodes tends to infinity. Equivalently, R_c is the minimal R such that $G(R)$ is a connected graph. It is well known [22] that in a homogeneous network R_c behaves as follows as a function of the number n of nodes, as n grows:

$$R_c = R_c(n) = \Big(\frac{V_n \log(n)}{n}\Big)^{1/d}. \tag{2}$$

2) Physical Model: We assume that all nodes choose a common power level P for all their transmissions. A transmission from a node X_i, $i \in \mathcal{T}$ is successfully received by a node X_j if

$$\text{SINR} = \frac{PG_{ij}}{N + \sum_{k\neq i, k\in\mathcal{T}} PG_{kj}} \geq \beta \tag{3}$$

where β is the threshold, N represents the ambient noise, and G_{ij} denotes the signal loss, whence PG_{ij} is the receiving power at the node j from the transmitter i. We assume a low power decay for the signal loss of the form $G_{ij} = |X_i - X_j|^{-\alpha}$, where $\alpha > d$ is the signal loss exponent. For a successful transmission the rate is assumed to be constant W.

We introduce the parameter $R_{\max}$ as the maximum possible distance between a transmitter and receiver to ensure a successful transmission under the Physical Model. Equation (3) implies that

$$R_{\max} = \Big(\frac{P}{N\beta}\Big)^{1/\alpha} \tag{4}$$

3) Generalized Physical Model: In this model the transmission rate W_{ij} between a sender i and a receiver j is determined using Shannon's formula for a wireless channel with additive Gaussian white noise [23].

$$W_{ij} = B\log_2\Big(1 + \frac{PG_{ij}}{BN_0 + \sum_{k\neq i,k\in\mathcal{T}} PG_{kj}}\Big) \tag{5}$$

where B is the bandwidth of the wireless channel and $N_0/2$ is the noise spectral density. While this model assigns a more realistic transmission rate at large distance, it also results in a singularity for the signal loss $G_{ij} = |X_i - X_j|^{-\alpha}$. Indeed, according to (5) the receiving power and the rate are amplified to unrealistic levels if sender and receiver are placed very closely. Some papers have pointed out this drawback [24], [25].

We introduce R_{m} as the maximum distance between any two nodes such that the packets can be received with rate B under the Generalized Physical Model. From (5) we have

$$R_{\text{m}} = \Big(\frac{P}{BN_0}\Big)^{1/\alpha} \tag{6}$$

D. Broadcast Capacity

We define the broadcast capacity for a subset $\mathcal{B} := \{Y_1, Y_2, \ldots\}$ of nodes that generate broadcast packets. Doing so adds flexibility and allows to cover cases where only a few nodes may be required to broadcast packets such as when using broadcast backbones or a subset of control enabled nodes in distributed applications.

Assume that the node Y_i generates packets at rate $\lambda_{Y_i} \geq 0$. We say that the rate vector $[\lambda_{Y_i}]_i$ is *achievable* if all nodes of the network receive all generated broadcast packets successfully within some given time $T_{\max} < \infty$. We study the maximum achievable broadcast rates for the case where pre-specified fractions of the aggregate broadcast rate are available to each node of $\mathcal{B}$. More precisely, given a vector of weights $\mathbf{g} = [g_i]_{i=1}^{\#\mathcal{B}}$, $g_i > 0$ such that $\sum_i g_i = 1$ we study the *broadcast capacity*

$$\lambda_{\mathcal{B}}(\mathbf{g}) := \sup\{a : \lambda_{Y_i} = g_i a, [\lambda_{Y_i}]_i \text{ is achievable}\} \tag{7}$$

Notably, the bounds we derive for the broadcast capacity $\lambda_{\mathcal{B}}(\mathbf{g})$ are independent of $\mathcal{B}$ and $\mathbf{g}$. In equivalent terms, the bounds we derive apply to both, $\overline{\lambda} := \sup_{\mathcal{B},\mathbf{g}} \lambda_{\mathcal{B}}(\mathbf{g})$ and $\underline{\lambda} := \inf_{\mathcal{B},\mathbf{g}} \lambda_{\mathcal{B}}(\mathbf{g})$ which are the maximum broadcast capacity which can be achieved by some nodes, respectively which can be achieved by any nodes.

IV. Broadcast Capacity under the Physical Model

In this section we compute bounds for the broadcast capacity of wireless networks using the Physical Model described in Section III.

A. Broadcast Capacity for Arbitrary Wireless Networks

Here, we determine upper and lower bounds for the broadcast capacity that apply to any arbitrary connected wireless network. The accuracy of these bounds varies with the network scenario.

We first note the hard upper bound W for the broadcast capacity which holds for any network. Indeed, since every node must receive the broadcast, its capacity cannot be higher than the maximum data rate at which a node can receive data.

We obtain the lower bound using the concept of the Minimum Connected Dominating Set (MCDS) of a network. To this end, note that the *network is interference-free connected* under the Physical Model exactly when the graph $G(R_{\max})$ is connected. Indeed, the maximal distance covered by a transmission is exactly $R_{\max}$. Finally, $\#\text{MCDS}(R_{\max})$ is well defined for an interference-free connected network, since $G(R_{\max})$ is connected. Broadcasting via exclusive transmissions over an MCDS we find

Theorem 1: Assume that the network is interference-free connected. Then, under the Physical Model,

$$\frac{W}{\#\text{MCDS}(R_{\max})+1} \leq \lambda_{\mathcal{B}}(\mathbf{g}) \leq W. \tag{8}$$

Instead of considering exclusive broadcasts, one may note that for $R < R_{\max}$ simultaneous transmissions on $G(R)$ become feasible. Assuming that $G(R)$ is connected Theorem 2 derives a bound on $\lambda_{\mathcal{B}}(\mathbf{g})$ for arbitrary topology by providing a TDMA scheduling method which considers the location of the nodes and which schedules the transmissions such as to reduce the interference between simultaneous transmissions. Allowing simultaneous transmissions Theorem 2 outperforms Theorem 1 for large networks.

Theorem 2: Assume that $G(R)$ is connected for some $R = R_{\max}/\rho$ with $\rho > 1$. Necessarily, the network is then interference-free connected. Then, under the Physical Model,

$$\frac{W}{K(\rho)} \leq \lambda_{\mathcal{B}}(\mathbf{g}) \tag{9}$$

where

$$K(\rho) = (5^d - 2^d)\left\lceil \sqrt{d}\Big(2 + \big(\frac{\beta \sum_{J\in\mathbb{Z}^d}^{|J|>0} |J|^{-\alpha}}{1-\rho^{-\alpha}}\big)^{1/\alpha}\Big)\right\rceil^d \tag{10}$$

Note that the explicit lower bound on $\lambda_{\mathcal{B}}(\mathbf{g})$ via (9) and (10) is improving as ρ grows. As the most important conclusion of theorems 1 and 2, the broadcast capacity is not affected by more than a constant factor when changing transmission power, number of nodes or volume V_n as long as for $R = R_{\max}/\rho$ the graph $G(R)$ remains connected with the same ρ.

B. Broadcast Capacity of Homogeneous Networks

As a corollary of Theorems 2 and 1 we are able to bound the broadcast capacity of a homogeneous network within a constant factor of W.

Theorem 3: Assume that the n nodes are uniformly distributed in a d-dimensional cube with volume V_n. Denote $\rho = \liminf_{n\to\infty} R_{\max}/R_c(n)$. If $\rho > 1$ then w.h.p. as $n \to \infty$

$$\frac{W}{K(\rho)} \leq \lambda_{\mathcal{B}}(\mathbf{g}) \leq W. \tag{11}$$

Clearly, if $\rho < 1$ then the network is disconnected w.h.p. and $\lambda_{\mathcal{B}}(\mathbf{g}) = 0$ for infinitely many n.

V. Broadcast Capacity under the Generalized Physical Model

As the main difference, in the Generalized Physical Model a node can transmit to any other node with some rate which depends on various parameters while in the other two channel models transmissions may or may not be successful. It does not come as a surprise that this model is harder to analyze and shows different behavior in some cases. We will show that poor signal strength and high interference are the two main limiting factors of the broadcast capacity. Clearly, already a single poorly connected node may form a bottleneck for the broadcast capacity. In well connected large networks, on the other hand, the broadcast capacity may suffer due to high interference caused by retransmissions.

Since the relative node locations have such a strong impact under this model, we obtain stronger results when making simple assumptions of a homogeneous networks, a situation often adopted in capacity studies.

A. Broadcast Capacity of Homogeneous Networks

We allow for power to be adjusted to the number of nodes, i.e., P may depend on n. Consequently, both $R_c = R_c(n)$ and R_m depend on n. We identify three asymptotic regimes as n grows by considering the rates of reception of the most isolated nodes to their next neighbors and to some given arbitrary node.

Low Power Regime: $P = O((V_n \log(n)/n)^{\alpha/d})$
In this case the transmission power is so limited that as n grows some *distant nodes* cannot receive within a constant of the bandwidth, even without interfering signals. These nodes then limit the broadcast capacity as follows

Theorem 4: Assume the nodes are uniformly distributed in d-dimensional cube with volume V_n. Assume there exists a constant c_1 such that $R_{\rm m} \leq c_1 R_c(n)$ for all n, i.e., $P \leq c_1^\alpha B N_0 (V_n \log(n)/n)^{\alpha/d}$. Then, w.h.p. as $n \to \infty$

$$\lambda_{\mathcal{B}}(\mathbf{g}) = \Theta\Big(B \log_2\big(1 + (\frac{R_{\rm m}}{R_c})^\alpha\big)\Big) \tag{12}$$

An example of a network setting in the Low Power Regime is the extended network model studied in [1] where $V_n = n$ with bounded transmission power. Then $\lambda_{\mathcal{B}}(\mathbf{g}) = \Theta(B \log_2(1 + \frac{P}{BN_0}\log(n)^{-\alpha/d}))$.

Tuned Power Regime: $P = \Omega((V_n \log(n)/n)^{\alpha/d})$ and $P = O(V_n^{\alpha/d})$
In this regime, the transmission power is sufficiently large so that every node can receive data with rate at least proportional to the bandwidth from some node. On the other hand, the area covered by the network is so large such that nodes cannot transmit their message to all nodes directly with such a rate. In this case the *interference* of simultaneous transmissions limits the capacity.

Theorem 5: Assume the nodes are uniformly distributed in d-dimensional cube with volume V_n. Assume there exists positive constants c_1 and c_2 such that $c_1 R_c(n) \leq R_{\rm m} \leq c_2 D$ for all n, i.e., $c_1^\alpha B N_0 (V_n \log(n)/n)^{\alpha/d} \leq P \leq c_2^\alpha d^{\alpha/2} B N_0 V_n^{\alpha/d}$. Then, w.h.p. as $n \to \infty$

$$\lambda_{\mathcal{B}}(\mathbf{g}) = \Theta(B) \tag{13}$$

An example of a network setting in the Tuned Power Regime is the dense network model where $V_n = 1$ and transmission power is bounded. Then, $\lambda_{\mathcal{B}}(\mathbf{g}) = \Theta(B)$. The broadcast capacity of this model has been left as open problem for the future study in [1].

High Power Regime: $P = \Omega(V_n^{\alpha/d})$
In this case the transmission power is so elevated that a node can transmit to all nodes directly with a rate that remains at least proportional to the bandwidth as n grows. In this regime, the area of the network decreases or the transmission power increases as number of nodes grows.

Theorem 6: Assume the nodes are uniformly distributed in d-dimensional cube with volume V_n. Assume there exits a constant c_2 such that $c_2 D \leq R_{\rm m}$ for all n, i.e., $P \geq c_2^\alpha d^{\alpha/2} B N_0 V_n^{\alpha/d}$. Then w.h.p. as $n \to \infty$

$$\lambda_{\mathcal{B}}(\mathbf{g}) = \Theta\Big(B \log_2\big(1 + (\frac{R_{\rm m}}{D})^\alpha\big)\Big) \tag{14}$$

This regime is usually not studied in the literature, as the assumptions seem unrealistic. We include it for completeness. Interestingly, Theorem 6 shows that in this regime a broadcast via multi-hop does not increase throughput by more than a constant factor as compared to a broadcast by one exclusive transmission.

VI. Conclusion and Future Work

We studied the broadcast capacity of wireless networks under two physically motivated channel models. First, in the Physical Model the broadcast capacity is shown to be within a constant of the channel capacity and little affected by the number of nodes or the volume occupied as long as the connectivity of network is maintained properly.

Second, in the Generalized Physical Model all nodes receive a broadcast at a rate depending on interference and distance to sender. Here, the broadcast capacity can vary significantly depending on how the transmission power is set in relation to the network size and topology. Focussing on homogeneous networks we identified three asymptotic regimes via explicit power settings.

Ongoing work concerns the capacity of wireless networks with more general topology.

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APPENDIX

Proof of Theorem 1: Consider an arbitrary node X_i in the network. The maximum rate of transmission or reception of data by X_i is W, since each broadcast packet must be either received or generated by X_i.

For showing the lower bound, we design a TDMA scheme which achieves it. We consider an MCDS of the graph $G(R_{\max})$ as a backbone. Every broadcast packet is transmitted from its generating node to a neighbor MCDS node, then the packet is retransmitted along MCDS nodes. We simply have only one transmission at the time, so clearly all nodes receive the packet successfully. This scheme gives us $W/(1+\#\mathrm{MCDS}(R_{\max}))$ throughput. ■

Proof of Theorem 2: Again, we design a TDMA scheme which achieves the lower bound. We do so in three steps.

Step 1: Divide the space into cells with diameter R such that the coordinates of their centers are $(i_1\frac{R}{\sqrt{d}}, i_2\frac{R}{\sqrt{d}}, \ldots, i_d\frac{R}{\sqrt{d}})$ for $i_1, \ldots, i_d \in \mathbb{Z}$ (see Fig. 1). Note that by design and by assumption any two nodes in the same cell are adjacent in $G(R)$. Next, we build a "cell graph" over the non-empty cells (colored grey in Fig. 1). The vertices of the cell graph are the non-empty cells and two cells are connected by an edge if there exist two nodes, one in each cell, that are adjacent in $G(R)$. Because $G(R)$ is connected it follows that the cell graph must be connected. We then build a spanning tree over the cell graph which we use to route broadcast packets.

Step 2: We assign color $L(r_{i_1}, r_{i_2}, \ldots, r_{i_d})$ to the cell with center $(i_1\frac{R}{\sqrt{d}}, \ldots, i_d\frac{R}{\sqrt{d}})$, where $r_i = i(\mathrm{mod})k$. The value of k is chosen large enough such that when two nodes in different cells with the same color transmit simultaneously, all of the nodes closer than R to the senders can receive successfully. We bound the value of k as follows. First, we find a lower bound for SINR for a given k. Recall that the simultaneous transmitters are in the different cells with the same color. For example, consider a transmitter Y in cell $(0, \ldots, 0)$ with a receiver X within distance R. Any simultaneous transmitter Z must lie in a cell $(j_1k, j_2k, \ldots, j_dk)$ where $J = (j_1, \ldots, j_d) \in \mathbb{Z}^d_\circ := \mathbb{Z}^d \backslash \{(0, \ldots, 0)\}$. The distance between the sender Z and the receiver X is then at least $\sqrt{j_1^2 + \ldots + j_d^2}\frac{kR}{\sqrt{d}} - 2R$, and since $|J| \geq 1$, at least $\geq R(\frac{k}{\sqrt{d}} - 2)|J|$. Thus,

$$\begin{aligned}\mathrm{SINR} &\geq \frac{PR^{-\alpha}}{N + \sum_{J\in\mathbb{Z}^d_\circ} P(R(\frac{k}{\sqrt{d}}-2)|J|)^{-\alpha}} \\ &= \frac{1}{\frac{N}{P}R^\alpha + (\frac{k}{\sqrt{d}}-2)^{-\alpha}\sum_{J\in\mathbb{Z}^d_\circ}|J|^{-\alpha}} \\ &= \frac{1}{\frac{1}{\rho^\alpha\beta} + (\frac{k}{\sqrt{d}}-2)^{-\alpha}\sum_{J\in\mathbb{Z}^d_\circ}|J|^{-\alpha}}\end{aligned}$$

One verifies now quickly that

$$k \geq \sqrt{d}((\frac{\beta\sum_{J\in\mathbb{Z}^d_\circ}\ |J|^{-\alpha}}{1-\rho^{-\alpha}})^{1/\alpha} + 2) \qquad (15)$$

ensures that the SINR becomes larger than β, guaranteeing successful simultaneous transmissions in cells of equal color. Notably, the lower bound of (15) is finite since $\alpha > d$.

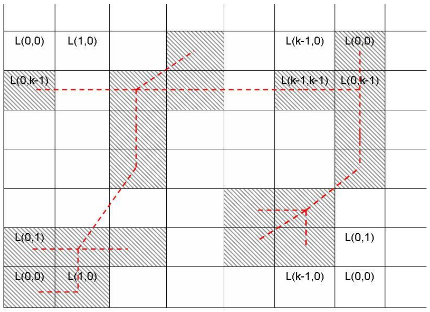

Fig. 1. Illustrating the collision free TDMA scheme for broadcasting in the case of a planar network. It uses k^2 colors to schedule the cells transmissions.

Step 3: For every pair of adjacent cells on the spanning tree of the cell graph we choose two nodes which connect the cells to be relays. When a packet needs to be forwarded from cell $S1$ to adjacent cell $S2$, the relay in cell $S1$ forwards the packet to the relay in cell $S2$. The relay in $S2$ rebroadcast the packet to all nodes in $S2$. If $S2$ is not a leaf vertex on the spanning tree of cell graph then the relays which connect it to other cells will forward the packet and the process continues till the broadcast packet has been disseminated to all nodes.

By geometry, it is easy to show that every cell has at most $5^d - 2^d - 1$ adjacent cells. There are thus at most $5^d - 2^d - 1$ relays in each cell. We divide the time slot corresponding to each color into $5^d - 2^d$ equal time slots of length T. In the first time slot corresponding to every color, each node $Y_i \in \mathcal{B}$ generates $WT\lambda_{Y_i}/((5^d - 2^d)k^d\lambda_{\mathcal{B}}(\mathbf{g}))$ bits for broadcast. If more than one Y_i's are located in the same cell they broadcast the packets sequentially in some order. In the remaining $5^d - 2^d - 1$ time slots of any particular color, the relays of cells with that color transmit (to other nodes in the cell or to the corresponding relay of an adjacent cell) any broadcast data that they have received but not yet forwarded.

Note that with this setup every relay can forward packets at the rate $W/(5^d - 2^d)k^d$ which establishes the desired lower bound for $\lambda_{\mathcal{B}}(\mathbf{g})$. ■

Proof of Theorem 3: The upper bound has been shown in Theorem 1. The lower bound follows quite easily from Theorem 2. Indeed, it is known for homogeneous networks [22] that the graph $G(R)$ is connected for large n w.h.p provided $R = (1+\epsilon)R_c$ for some $\epsilon > 0$. Choosing ϵ small enough such that $\frac{\rho}{1+\epsilon} > 1$, Theorem 2 provides the lower bound $W/K(\frac{\rho}{1+\epsilon})$. Now, letting $\epsilon \to 0$, by left continuity of $K(\rho)$ the lower bound becomes $W/K(\rho)$. ■

Proof of Theorem 4: We establish the upper bound the following lemma.

Lemma 1: In a homogeneous network

$$\lambda_{\mathcal{B}}(\mathbf{g}) \le B\log_2(1 + (\frac{R_{\rm m}}{R_c})^{\alpha}) \tag{16}$$

w.h.p. for large n.

Proof: It has been proved in [22] that for any $0 < a < 1$, $G(aR_c)$ contains $\Omega(n^{1-a^d})$ isolated nodes w.h.p. for large n. Consider an isolated node, the receiving rate for the node is bounded by $B\log_2(1+P(aR_c)^{-\alpha}/BN_0)$. This gives us

$$\lambda_{\mathcal{B}}(\mathbf{g}) \le B\log_2(1 + \frac{P(aR_c)^{-\alpha}}{BN_0}) = B\log_2(1 + (\frac{R_{\rm m}}{aR_c})^{\alpha}) \tag{17}$$

Then, letting $a \to 1$ yields (17). ♢

Next, we show a constant factor of the computed upper bound is achievable. We divide the volume V_n into cube cells with diameter bR_c for a constant $b > 1$ (e.g. $b = 2$). From [22] we know that $G(bR_c)$ is connected w.h.p. for large n. Then we apply the broadcast scheme developed in the proof of Theorem 2. We may set the parameter k to any integer larger than 3, i.e. here we choose $k = 4$. Then the SINR of a received broadcast packets becomes

$$\begin{aligned} \text{SINR} &\ge \frac{1}{\frac{BN_0}{P}(bR_c)^{\alpha} + (\frac{4}{\sqrt{d}} - 2)^{-\alpha}\sum_{J\in\mathbb{Z}_{\circ}^d}|J|^{-\alpha}} \\ &= \frac{R_{\rm m}^{\alpha}(bR_c)^{-\alpha}}{1 + R_{\rm m}^{\alpha}(bR_c)^{-\alpha}(\frac{4}{\sqrt{d}} - 2)^{-\alpha}\sum_{J\in\mathbb{Z}_{\circ}^d}|J|^{-\alpha}} \\ &\ge \frac{1}{1 + (c_1/b)^{\alpha}(\frac{4}{\sqrt{d}} - 2)^{-\alpha}\sum_{J\in\mathbb{Z}_{\circ}^d}|J|^{-\alpha}}(\frac{R_m}{bR_c})^{\alpha} \end{aligned}$$

Therefore, the throughput of this scheme is at least $\Theta(B\log_2(1 + (\frac{R_{\rm m}}{R_c})^{\alpha}))$. ■

Proof of Theorem 5: First we show that $\lambda_{\mathcal{B}}(\mathbf{g}) = \Theta(B)$ is achievable. We broadcast the packets similarly to the proof of Theorem 4. Here, the SINR of broadcast packets is at least

$$\begin{aligned} \text{SINR} &\ge \frac{1}{\frac{BN_0}{P}(bR_c)^{\alpha} + (\frac{4}{\sqrt{d}} - 2)^{-\alpha}\sum_{J\in\mathbb{Z}_{\circ}^d}|J|^{-\alpha}} \\ &= \frac{1}{R_{\rm m}^{-\alpha}(bR_c)^{\alpha} + (\frac{4}{\sqrt{d}} - 2)^{-\alpha}\sum_{J\in\mathbb{Z}_{\circ}^d}|J|^{-\alpha}} \\ &\ge \frac{1}{(b/c_1)^{\alpha} + (\frac{4}{\sqrt{d}} - 2)^{-\alpha}\sum_{J\in\mathbb{Z}_{\circ}^d}|J|^{-\alpha}} \end{aligned}$$

So, the throughput in this case is at least $\Theta(B)$.

To prove the upper bound we consider the set $\mathcal{I}$ of isolated nodes of $G(R)$ where $R = a\sqrt[d]{V_n/n}$ and a is a fixed constant. In the following lemmas we establish two properties. First, there exists $p > 0$ such that $\#\mathcal{I} \ge p \cdot n$ for large n and w.h.p. (see lemma 2).

Second, the nodes in $\mathcal{I}$ are so far from all other nodes, that the sum of receiving rates is always bounded by $K \cdot B\#\mathcal{I}$ where K is a constant number and B is the bandwidth (see lemma 3). Consequently, since the sum of receiving rates of nodes of $\mathcal{I}$ must be at least $\lambda_{\mathcal{B}}(\mathbf{g})\#\mathcal{I}$ on average we find $\lambda_{\mathcal{B}}(\mathbf{g}) \leq KB = \Theta(B)$ w.h.p. ■

Lemma 2: For some $p > 0$, $P[\#\mathcal{I} > np] \to 1$ as $n \to \infty$.

Proof: While we establish the lemma for homogeneous networks, we mention that a proof for the Poisson model is easy to come by with.

Assume that n nodes are uniformly distributed in V_n. To each node X_i we assign a random value ν_i, where ν_i is 1 if X_i is isolated in $G(R)$, and 0 otherwise. Thus, the random variable $S = \sum_{i=1}^{n} \nu_i$ represents number of isolated nodes. We also define V'_n to be the set of points in V_n which are within distance less than R from the boundary of V_n. In abuse of notation we denote its volume also by V'_n. Since $R = O((\frac{V_n}{n})^{1/d})$, it follows that $V'_n = O(\frac{V_n}{n})$.

We compute $\mathbb{E}(\nu_i) = P[\nu_i = 1]$ as follows

$$\begin{aligned}\mathbb{E}(\nu_i) &= P[\nu_i = 1|X_i \notin V'_n]P[X_i \notin V'_n] \\ &\quad + P[\nu_i = 1|X_i \in V'_n]P[X_i \in V'_n] \\ &= (1 - \frac{\pi_d R^d}{V_n})^{n-1}(1 - O(\frac{1}{n})) + O(\frac{1}{n}) \\ &= \exp(-\pi_d a^d) + O(\frac{1}{n})\end{aligned}$$

where π_d is the volume of unit radius sphere in d-dimensional space. Also, $\mathbb{E}(\nu_i\nu_j) = P[\nu_i = 1, \nu_j = 1]$ becomes

$$\begin{aligned}\mathbb{E}(\nu_i\nu_j) &= P[\nu_i = \nu_j = 1|X_i, X_j \notin V'_n]P[X_i, X_j \notin V'_n] \\ &\quad + P[\nu_i = \nu_j = 1|X_i \text{ or } X_j \in V'_n]P[X_i \text{ or } X_j \in V'_n] \\ &= P[\nu_i = \nu_j = 1|X_i, X_j \notin V'_n] + O(\frac{1}{n})\end{aligned}$$

Splitting the two cases $|X_i - X_j| > 2R$ and $|X_i - X_j| \leq 2R$

$$\begin{aligned}\mathbb{E}(\nu_i\nu_j) &= P[\nu_i = \nu_j = 1|X_i, X_j \notin V'_n, |X_i - X_j| > 2R] \\ &\quad \cdot P[|X_i - X_j| > 2R]] + O(\frac{1}{n}) \\ &= (1 - \frac{\pi_d R^d}{V_n})^{n-2}(1 - \frac{\pi_d R^d}{V_n})^{n-2} + O(\frac{1}{n}) \\ &= \exp(-2\pi_d a^d) + O(\frac{1}{n})\end{aligned}$$

Setting $p = \exp(-\pi_d a^d)/2$ we have $\mathbb{E}(S) = 2pn + O(1)$ and $\text{var}(S) = \mathbb{E}(S^2) - (\mathbb{E}(S))^2 = O(n)$. Finally,

$$P[|S - \mathbb{E}(S)| > \mathbb{E}(S)/2] \leq \frac{\text{var}(S)}{(\mathbb{E}(S)/2)^2} = \frac{O(n)}{\Theta(n^2)} \qquad (18)$$

by the Markov inequality [26]. ◇

Lemma 3: The sum of receiving rates of the nodes of $\mathcal{I}$ is bounded by $K \cdot B \cdot \#\mathcal{I}$, where K is a constant.

For the proof, we consider two cases.

Case (1): Assume that X_s is the only transmitter in the network. From equation (5) we have

$$\begin{aligned}\sum_{i\in\mathcal{I}} W_{si} &\leq \sum_{i\in\mathcal{I}} B\log_2(1 + \frac{P|X_i - X_s|^{-\alpha}}{BN_0}) \\ &\leq \sum_{k=1}^{\lceil D/R-1\rceil} f(k)B\log_2(1 + \frac{P(kR)^{-\alpha}}{BN_0})\end{aligned}$$

where $f(k)$ is the maximum number of the isolated nodes in distance l from X_s where $kR < l \leq (k+1)R$ (note that $|X_i - X_s| > R$ so we start from $k = 1$). Now consider the balls with radius $R/2$ around nodes of $\mathcal{I}$. Since the nodes are in distance at least R from each other, the balls are disjoint and by simple geometric argument we bound the number of balls by $f(k) < \frac{\pi_d(k+1+.5)^d R^d - \pi_d(k-.5)^d R^d}{\pi_d(.5R)^d} = (k+3)^d - (k-1)^d$. So, clearly there is a constant K_1 such that $f(k) < K_1 k^{d-1}$. Then, continuing and applying Cauchy-Schwartz

$$\begin{aligned}\sum_{i\in\mathcal{I}} W_{si} &\leq \sum_{k=1}^{\lceil D/R-1\rceil} K_1 k^{d-1} B\log_2(1 + (\frac{R_{\text{m}}}{kR})^\alpha) \\ &\leq \sum_{k=1}^{\lceil D/R-1\rceil} K_1 k^{d-1} B\log_2(\frac{D^\alpha + R_{\text{m}}^\alpha}{k^\alpha R^\alpha}) \\ &\leq K_1 B\sqrt{\sum_{k=1}^{\lceil D/R-1\rceil} k^{2d-2} \sum_{k=1}^{\lceil D/R-1\rceil} \log_2^2(\frac{D^\alpha + R_{\text{m}}^\alpha}{k^\alpha R^\alpha})} \\ &\leq K_1 B\sqrt{K_2(\frac{D}{R})^{2d-1}\int_0^{D/R} \log_2^2(\frac{D^\alpha + R_{\text{m}}^\alpha}{x^\alpha R^\alpha})dx}\end{aligned}$$

$$\begin{aligned}&= K_1 B\sqrt{K_2(\frac{D}{R})^{2d-1}} \\ &\quad \cdot\sqrt{\frac{D}{R}[2\alpha^2 + 2\ln(\frac{D^\alpha + R_{\text{m}}^\alpha}{D^\alpha}) + \ln^2(\frac{D^\alpha + R_{\text{m}}^\alpha}{D^\alpha})]\log_2(e)} \\ &\leq K_1 B\sqrt{K_2(\frac{D}{R})^{2d-1}K_3(\frac{D}{R})} = BK_1\sqrt{K_2K_3}(\frac{D}{R})^d \\ &= K_1\sqrt{K_2K_3 d}Bn/a^d \leq K_1\sqrt{K_2K_3 d}B\#\mathcal{I}/(pa^d)\end{aligned}$$

where the constant K_2 depends on d, and constant K_3 depends on c_2. Since this is a bound on the sum of receiving rates of nodes of $\mathcal{I}$, we result that in case (1) $\lambda_{\mathcal{B}}(\mathbf{g}) \leq K_1\sqrt{K_2K_3 d}\frac{1}{pa^d}B$

Case (2): Now assume that there are more than one simultaneous transmitters. Denote $\mathcal{T}$ the set of transmitters. For every $X_s \in \mathcal{T}$, we consider $X_{s'}$ as the closest transmitter to X_s and we define $U(s) = |X_s - X_{s'}|/2$. Also we define $\text{Ball}(s)$ to be the ball with radius $U(s)$ around the node X_s. In lemma 4 we show that these balls are disjoint. Moreover, lemma 5 bounds the receiving rate of the nodes in terms of the radii of the balls. ◇

Lemma 4: The balls $\text{Ball}(s)$ *($X_s \in \mathcal{T}$) are disjoint in d-dimensional space.*

Proof: Assume that two balls $\text{Ball}(s_1)$ and $\text{Ball}(s_2)$ are not disjoint and $U(s_1) \leq U(s_2)$. Then, $|X_{s_2} - X_{s_1}| < U(s_1) + U(s_2) \leq 2U(s_2)$. It shows that X_{s_1} is closer than the closest transmitter to X_{s_2}. This is a contradiction. $\diamondsuit$

Lemma 5: Assume X_s the closest transmitter to an arbitrary node X_i, then the receiving rate of X_i is bounded by $B[1+\alpha\log_2(1+\frac{2U(s)}{|X_i-X_s|})]$. As a result if X_i is not located in the $Ball(s)$, the rate is bounded by $[1+\alpha\log_2(3)]B$.

Proof: The receiving rate of X_i is bounded by W_{si} because the signal of X_s is the strongest receiving signal at X_i. So, we bound W_{si} as the following

$$\begin{aligned} W_{si} &\leq B\log_2(1+\frac{P|X_i-X_s|^{-\alpha}}{BN_0+P|X_i-X_{s'}|^{-\alpha}}) \\ &\leq B\log_2(1+(\frac{|X_i-X_{s'}|}{|X_i-X_s|})^{\alpha}) \\ &\leq B\log_2(1+(\frac{|X_i-X_s|+2U(s)}{|X_i-X_s|})^{\alpha}) \\ &\leq B\log_2(2(1+\frac{2U(s)}{|X_i-X_s|})^{\alpha}) \\ &= B[1+\alpha\log_2(1+\frac{2U(s)}{|X_i-X_s|})] \quad \diamondsuit \end{aligned}$$

Next, we bound the sum of receiving rates for the nodes of $\mathcal{I}$ inside $\text{Ball}(s)$ for an arbitrary $X_s \in \mathcal{T}$.

$$\begin{aligned} \sum_{i\in\text{Ball}(s)} W_{si} &\leq \sum_{i\in\text{Ball}(s)} B[1+\alpha\log_2(1+\frac{2U(s)}{|X_i-X_s|})] \\ &\leq \sum_{k=1}^{\lceil U(s)/R-1\rceil} f(k)B[1+\alpha\log_2(1+\frac{2U(s)}{kR})] \\ &\leq K_1B\sum_{k=1}^{\lceil U(s)/R-1\rceil} k^{d-1}[1+\alpha\log_2(\frac{3U(s)}{kR})] \end{aligned}$$

by applying Cauchy-Schwarz inequality we have

$$\begin{aligned} &\leq K_1B\sqrt{\sum_{k=1}^{\lceil U(s)/R-1\rceil} k^{2d-2}\sum_{k=1}^{\lceil U(s)/R-1\rceil}[1+\alpha\log_2(\frac{3U(s)}{kR})]^2} \\ &\leq K_1B\sqrt{K_2(\frac{U(s)}{R})^{2d-1}\int_0^{U(s)/R}[1+\alpha\log_2(\frac{3U(s)}{xR})]^2dx} \\ &\leq K_1B\sqrt{K_2(\frac{U(s)}{R})^{2d-1}K_4(\frac{U(s)}{R})} \\ &= BK_1\sqrt{K_2K_4}(\frac{U(s)}{R})^d \end{aligned}$$

where K_4 is a constant which is resulted from the integral.

From the lemma 4 the balls $\text{Ball}(s)$ ($X_s \in \mathcal{T}$) are disjoint. On the other hand, we can easily show that all the balls are placed inside the cube $\overline{V}_n$ of side size $(2\sqrt{d}+1)\sqrt[d]{V_n}$ and where V_n is located in its center. Therefore, we have

$$\sum_{s\in\mathcal{T}}\pi_d U(s)^d \leq (2\sqrt{d}+1)^d V_n \tag{19}$$

Combining this equation with the computed upper bound

$$\begin{aligned} \sum_{s\in\mathcal{T}}\sum_{i\in\text{Ball}(s)} W_{si} &\leq BK_1\sqrt{K_2K_4}\frac{(2\sqrt{d}+1)^d}{\pi_d}\frac{V_n}{R^d} \\ &= K_1\sqrt{K_2K_4}\frac{(2\sqrt{d}+1)^d}{\pi_d a^d}Bn \end{aligned}$$

Then, we bound the sum of receiving rates of the nodes of $\mathcal{I}$ as follows

$$\begin{aligned} \sum_{i\in\mathcal{I}} W_{si} &= \sum_{s\in\mathcal{T}}\sum_{i\in\text{Ball}(s)} W_{si} + \sum_{i\notin\bigcup\text{Ball}(s)} W_{si} \\ &< K_1\sqrt{K_2K_4}\frac{(2\sqrt{d}+1)^d}{\pi_d a^d}Bn + [1+\alpha\log_2(3)]B\#\mathcal{I} \\ &\leq K_1\sqrt{K_2K_4}\frac{(2\sqrt{d}+1)^d}{\pi_d p a^d}B\#\mathcal{I} + [1+\alpha\log_2(3)]B\#\mathcal{I} \end{aligned}$$

Therefore, in case(2) we have $\lambda_{\mathcal{B}}(\mathbf{g}) < [K_1\sqrt{K_2K_4}(2\sqrt{d}+1)^d/(\pi_d p a^d) + 1 + \alpha\log_2(3)]B$ ■

Proof of Theorem 6: The following lemma provides a lower bound.

Lemma 6: Assume the network is contained in a bounded set with diameter D, then

$$B\log_2(1+(\frac{R_{\text{m}}}{D})^{\alpha}) \leq \lambda_{\mathcal{B}}(\mathbf{g}) \tag{20}$$

Proof: If the source node of broadcast transmit the packets to all nodes directly, then every node in the network can receive with the rate larger than $B\log_2(1+\frac{PD^{-\alpha}}{BN_0}) = B\log_2(1+(\frac{R_{\text{m}}}{D})^{\alpha})$, because the distance between the source and any node is less than D. $\diamondsuit$

For the proof of upper bound, we build a set $\mathcal{I}$ similar to the proof of Theorem 5 and we consider the analogous two cases. In case (1), we follow the same inequalities, but we write $K_5\log_2^2(1+(\frac{R_{\text{m}}}{D})^{\alpha})$ (where K_5 is constant depends on c_2) instead of K_3. Continuing the argument we find that $\lambda_{\mathcal{B}}(\mathbf{g})$ is bounded by $\Theta(B\log_2(1+\frac{R_{\text{m}}}{D})^{\alpha})$. In case (2), we use the same inequalities as before finding that $\lambda_{\mathcal{B}}(\mathbf{g})$ is bounded by $\Theta(B)$. This shows that using more than one simultaneous transmitters might even decrease the broadcast throughput in the third regime. ■