Brownian motion

Importance and History

Brownian motion, which was observed as early as in 1765, but was first studied by Robert Brown in 1827 for pollen grains in water, denotes the erratic motion (or models thereof) of microscopic particles suspended in a liquid or gas. In the late nineteenth century, studies attempted to explain Brownian motion as a consequence of thermal motion of molecules in the liquid colliding with the larger particles thereby causing their apparent random erratic motion. However, Brownian motion was not widely known until 1905 when Albert Einstein made precise predictions on how the random impacts of molecules would result in an irregular motion of larger particles. His predictions allowed inference of the number and speed of the molecules hitting the particle. His work was also used to compute the size of atoms from observing the Brownian motion of a particle, thus removing any remaining doubt about the existence of atoms at that time.

Today, Brownian motion owes its relevance not only to its physical interpretation as the basic ingredient to diffusion theory but also to its particular statistical properties that rendered it useful in environmental studies [1, 2], biology [3], economics, computational finance, queueing theory, and statistics, among others. Visualizations are abundant online.

Mathematical Model

The path of a particle undergoing Brownian motion in a d-dimensional space ($d \geq 2$) is modeled by a stochastic process called the *Wiener process* whose coordinates form mutually independent one-dimensional Brownian motions. Thus, we start with the simpler case $d = 1$ and return to the general case only toward the end. For simplicity, we only consider nonnegative times, that is, $t \geq 0$.

Let $\mathcal{N}(\mu, \sigma^2)$ denote the normal distribution (*see* **Normal and multinormal distribution**) with expected value μ and variance σ^2. A stochastic process $W(t)$ (*see* **Stochastic process**) with the following three properties is called a *Wiener process* or *standard Brownian motion*

1. $W(0) = 0$ almost surely (that is, with probability one, almost surely, a.s.).
2. $W(t)$ is continuous a.s. (as a function of t).
3. W_t has independent increments with distribution $W(t) - W(s) \sim \mathcal{N}(0, |t - s|)$.

Norbert Wiener showed in 1923 that such a process exists and discovered many of its properties, whence it was named in his honor (see Ref. 4). The process $W_x(t) = W(t) + x$ starts at the point x a.s. instead of 0 but shares properties (2) and (3). Also, the process $\sigma W(t)$ is sometimes called *(ordinary) Brownian motion*. For simplicity, this article considers only standard Brownian motion, which can be obtained from any ordinary Brownian motion $X(t)$ by normalizing it via $W(t) = cX(t)$, where c is such that $\mathrm{var}(W(1)) = 1$.

Having "independent increments" means that $W(v) - W(u)$ and $W(t) - W(s)$ are independent random variables whenever $0 \leq s \leq t \leq u \leq v$. In other words, $W(t)$ is a Lévy process (*see* **Lévy processes**). On the other hand, any Lévy process can be thought of as the sum of three components: a drift (the deterministic part), a Brownian motion (the diffusion part), and a jump component (the discontinuous part).

Alternatively, the following characterizations identify a process as being a Wiener process:

- **Statistical.** Brownian motion is the unique Gaussian process (*see* **Gaussian process**) with zero mean and autocovariance function

$$\mathbb{E}[W(s)W(t)] = \min(s, t)$$

Note: It is always possible to choose a continuous version of such a Gaussian process. Also, the autocovariance function implies that Brownian motion is short-range dependent. Long-range dependent Gaussian processes exist under the name *fractional Brownian motion* (*see* **Long-range dependence** and Ref. [5, sect. 7.2]).

- **Lévy Characterization.** A Wiener process is an almost surely continuous process $W(t)$ with $W(0) = 0$ and such that both $W(t)$ and $W(t)^2 - t$ are martingales (with natural filtration) (see [6] Ref. [9, thm. 8.6.1.]).

Paths of a Wiener process can be constructed in the following ways:

- **Midpoint displacement.** Iterating in n, we construct interpolation values at times $t = k/2^n$ ($k =$

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$0, \ldots, 2^n$) and denote the linear interpolation at stage n by $W_n(t)$. We start at $n = 0$ with $W_0(0) = 0$ and $W_0(1) \sim \mathcal{N}(0, 1)$. Iteratively, given $W_n(t)$ we leave the interpolation values at $t = k/2^n$ as they are and add at $t = (2k + 1)/2^{n+1}$ the values $(W_n(k/2^n) + W_n((k + 1)/2^n)/2 + Z_{k,n}$ where the $Z_{k,n}$ are independent $\mathcal{N}(0, 1/2^{n+1})$. With probability one, the piecewise linear paths of the W_n converge uniformly to Brownian motion (see Ref. [6, sect. 6.3.]). This method has the advantage of allowing the refinement of a discrete approximation of a Brownian path at any stage and at any location.

- **Rescaling.** Take independent random variables $Z_k \sim \mathcal{N}(0, 1)$, let $S_k = Z_1 + \cdots + Z_k$ and let $W_0(t)$ be the linear interpolation of the points (k, S_k). Then the processes $W_n(t) = W_0(nt)/\sqrt{n}$ possess interpolation points at $t = k/n$ and converge in full distribution to a Wiener process as n goes to infinity. This construction is called the *invariance principle* and has far reaching consequences in statistical estimation via Donsker's theorem (see Ref. [7, p. 66] or Ref. [6, sect. 6.1 and 6.10]).

Properties

Uniqueness

The Wiener process is unique in the sense of finite dimensional properties. This is a direct consequence of the fact that the joint Gaussian distributions are fully determined by their means and pairwise covariances.

However, it is not pathwise unique. As the next paragraph illustrates, there are several ways to construct new Wiener processes from an existing one. Though pathwise different, these processes share identical distributions. Furthermore, one can always construct families of independent, thus pathwise different Wiener processes. This becomes relevant, for example, in the context of diffusion processes and most importantly when constructing Brownian motion in higher dimensions.

Symmetry, Restart, Time inversion, Rescaling

If $W(t)$ is a Wiener process, then $-W(t)$, $\tilde{W}_s(t) = W(s + t) - W(s)$, $tW(1/t)$ and $\sqrt{a}W(t/a)$ are Wiener processes as well.

Stopping Times

A random time T is called a *stopping time* if at time t it can be decided whether the time instance T has already occurred by looking only at the past until t but not at the future of the Wiener process from t on. Technically, T is a stopping time if the event $\{T \leq t\}$ lies in the sigma field generated by the Wiener process up to time t (compare **Waiting time**).

The first time that a Wiener process $W(t)$ assumes the value a is a stopping time; it is called *hitting time* and will be denoted by T_a. The first time that a Wiener process reaches its maximum over the time interval $[0, 1]$ is not a stopping time.

Strong Markov Property

The increments of a Wiener process are *strongly independent*, meaning that the process restarted at a stopping time T, thus, the process $\tilde{W}_T$ is again a Wiener process and independent from the original Wiener process until time T. Consequently, a Wiener process possesses the *strong Markov property* (*see* **Markov chains**, **Markov process**, **Markov random field models**) meaning that predicting a state after a stopping time using the entire history until the stopping time is equivalent to using only the state at the stopping time itself.

Reflection Principle

The process

$$W^*(t) = \begin{cases} W(t) & \text{for} \quad 0 \leq t \leq T_a \\ 2a - W(t) & \text{for} \quad t > T_a \end{cases}$$

can be thought of as reflecting the original Wiener process W at $y = a$ from T_a on. Because of the strong Markov property, the reflected process W^* is a Wiener process as well.

Maximum

Denote by $M(t)$ the maximal value of a Wiener process up to time t: $M(t) = \sup\{W(s) : 0 \leq s \leq t\}$ (*see* **Exceedance probability**, **Extreme value analysis**). Then, the reflection principle implies the surprisingly simple fact that $M(t)$ has the same

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distribution as $|W(t)|$. The distribution function of T_a follows from Refs [6, sect. 6.5; 7, p. 96]

$$P[T_a \le t] = P[M(t) \ge a] = \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{\infty} \mathrm{e}^{-\frac{x^2}{2}}\, \mathrm{d}x \tag{1}$$

with density $\frac{a}{\sqrt{2\pi t^3}} \exp(-\frac{a^2}{2t})$. Note that T_a is a.s. finite, but its mean is infinite: $\mathbb{E}[T_a] = \infty$.

Instant Reflection

The *instantly reflected Brownian motion* is $|W(t)|$; it is a Markov process, and has the same finite dimensional distributions as $M(t) - W(t)$ (see Ref. [7, p. 97]).

Escape from a Strip

Let a and b be strictly positive numbers. Let $T_{a,b}$ denote the first time at which a Wiener process hits the outside of the open interval $(-a, b)$. Its mean is $\mathbb{E}[T_{a,b}] = ab$, and its Laplace transform is

$$\mathbb{E}[\mathrm{e}^{-\theta T_{a,b}}] = \frac{\sinh(a\sqrt{2\theta}) + \sinh(b\sqrt{2\theta})}{\sinh((a+b)\sqrt{2\theta})}$$

(see Ref. [6, sect. 6.7]). The odds for exiting through $-a$ are b/a:

$$P[W(T_{a,b}) = -a] = \frac{b}{a+b}$$
$$P[W(T_{a,b}) = b] = \frac{a}{a+b}$$

For $0 < a < b$, the time from the first hit at a to the first hit at b is $T_b - T_a$; owing to strong Markov it is independent of the past and has the same distribution as T_{b-a} (see Ref. [7, prop. 8.5]).

Nonstopping Simes and Zeros

Denote by U_t the last time in the interval $[0, t]$ where W attains its maximum value $M(t)$. The distribution of U_t is $P[U_t \le s] = \frac{2}{\pi} \arcsin \sqrt{\frac{s}{t}}$. The first time that W attains its maximum $M(t)$ has the same distribution as U_t, and thus, the Wiener process assumes the value $M(t)$ a.s. only once in the interval $[0, t]$ (see Ref. [7, rem. 8.16]). Also the last time at which W assumes the value 0 in the interval $[0, t]$ follows the same distribution as U_t.

Path Oscillations

Almost surely, the paths and graphs are continuous but they oscillate widely and are nowhere differentiable, a.s. Indeed, the so-called Lévy modulus of continuity states that a.s. roughly $|W(t) - W(s)| \simeq A|t - s|^{1/2}$ everywhere. More precisely, almost surely (see Ref. [8, thm. 10.2])

$$\limsup_{h \to 0^+} \frac{\sup_{0 \le t \le 1} W(t+h) - W(t)}{\sqrt{2h \log(1/h)}} = 1 \tag{2}$$

or, by Khintchine's law of the iterated logarithm, we have for any t, a.s. (see Ref. [6, p. 546])

$$\limsup_{h \to 0^+} \frac{W(h)}{\sqrt{2h \log\log(1/h)}} = 1$$
$$\liminf_{h \to 0^+} \frac{W(h)}{\sqrt{2h \log\log(1/h)}} = -1$$

Time inversion reveals the a.s. long-term behavior

$$\limsup_{t \to \infty} \frac{W(t)}{\sqrt{2t \log\log(t)}} = 1$$
$$\liminf_{t \to \infty} \frac{W(t)}{\sqrt{2t \log\log(t)}} = -1.$$

Further facts relating to the strong oscillations are as follows: the set of times at which a Wiener process attains a *local* maximum is countable and dense. The Wiener process is monotone in no interval, a.s. The "Zero set" $Z = \{t \ge 0 : W(t) = 0\}$ is uncountable and contains no isolated points; in other words, whenever the Wiener process passes through zero, there will be arbitrarily many time instances arbitrarily close at which the Wiener process passes through zero as well. Moreover, Z has Hausdorff dimension $1/2$.

Another measure of the strength of the oscillations of a Brownian path is the fact that the Hausdorff dimension of the graph of Brownian motion equals $3/2$.

Stationarity

Clearly, Brownian motion itself is not stationary since $\mathrm{var}(W(t)) = t$. But $X(t) = \mathrm{e}^{-t/2} W(\mathrm{e}^t)$ is an

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Ornstein–Uhlenbeck process and is stationary with mean zero and autocovariance (see Ref. [6, p. 556] and Ref. [5, p. 313]

$$\mathbb{E}[X(t)X(s)] = \mathrm{e}^{|t-s|/2}$$

The increment process $I_h(t) = W(t+h) - W(t)$ with fixed lag $h > 0$ (not to be confused with the restarted process $\tilde{W}$) is stationary with zero mean and autocovariance $\mathbb{E}[I_h(x+t)I_h(x)] = h - \min(h, t)$.

Gaussian White Noise

In loose terms, Gaussian white noise is the "derivative" of Brownian motion. However, Brownian paths possess no derivatives, anywhere. A short overview of approaches that attempt to make this interpretation meaningful follows.

For any $h > 0$, the sequence $G_k = I_h(kh)$ forms a time series of i.i.d. $\mathcal{N}(0, h)$ random variables, called *discrete Gaussian white noise*. It constitutes the most basic random walk (*see* **Random walk**). Clearly, $W(nh) = G_0 + \cdots + G_{n-1}$. A continuous-time version of this relation would use the derivative of W, which does not exist (see above). Still, it is possible to define integration with respect to a Wiener process $\int f(s)\mathrm{dW(s)}$ in the sense that

$$W(t) \overset{m.s.}{=} \int_0^t \mathrm{d}W(s)$$

However, equality is not pathwise, but in a mean square sense (see Refs [7, 9]).

The framework of "mean square calculus" is a different approach, being concerned with describing properties of second-order stationary processes through auto- and cross-correlation functions (see Ref. [10, p. 384 ff]). While correlations provide a weak description of processes, they can still capture the finite dimensional distributions of Gaussian processes (*see* **Gaussian process**). Even within this framework, a mean-square-derivative of the Wiener process does not exist (see Ref. [10 p. 426 ff]). But a *continuous-time Gaussian white noise* can be defined as the degenerate mean-square-limit $G(t) = \lim_{m.s.} I_h(t)/h$ with autocorrelation function equal to the Dirac distribution and spectral density constant equal to 1 in the distributional sense (*see* **Spectral methods**). This spectral representation gave raise to the name "white" noise since all frequencies are present with equal amplitude, as in (ideal) white light.

In the mean square sense, the Kahunen–Löwe expansions of Wiener process and Gaussian white noise for $0 \le t \le 1$ are, using i.i.d. $\mathcal{N}(0, 1)$ random variables C_n and D_n,

$$W(t) \overset{m.s.}{=} \sum_{n=1}^{\infty} C_n \frac{\sqrt{2}}{(n-1/2)\pi} \sin\left((n-1/2)\pi t\right)$$

$$G(t) \overset{m.s.}{=} \sum_n D_n \Phi_n(t)$$

where Φ_n is any mean-square orthonormal basis such as the Fourier basis or an appropriate wavelet basis. Choosing the Haar wavelet one rediscovers the midpoint displacement.

Advanced Properties

We mention further properties with less detail. Denote the maximal modulus up to time $t = 1$ by $A = \sup(|W(t)| : 0 \le t \le 1)$. A bound for the tail distribution of A is given by (see Ref. [11, Lemma 12.2.1, p. 219])

$$P[A > a] \le 8\exp(-ca^2)$$

where c is some constant. Denote the maximal value of a Brownian motion with a drift μ by $M_\mu = \sup_{t>0}(W(t) - \mu t)$. For $\mu < 0$, M_μ is exponentially distributed meaning that for $a > 0$ (compare with (1))

$$P[M_\mu > a] = \mathrm{e}^{-2|\mu|a}$$

A Brownian bridge is a process $B(t) = W(t) - tW(1)$, where W is a standard Brownian motion. The Brownian bridge returns to zero at time $t = 1$ by construction. Its Kahunen–Löwe expansion reads as

$$B(t) \overset{m.s.}{=} \sum_{n=1}^{\infty} H_n \frac{\sqrt{2}}{n\pi} \sin(n\pi t)$$

where the H_n denote i.i.d. $\mathcal{N}(0, 1)$ random variables. The Brownian bridge is of particular importance in statistics in the context of Donsker's theorem and the Kolmogorov–Smirnov test, which is useful in testing whether samples come from a given probability distribution (*see* **Kernel density estimation** and Ref. 6, sect. 6.10).

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Brownian Motion in Higher Dimensions

Intersections

Consider the question whether several Brownian motions starting at different points in some d-dimensional space intersect, meaning that there is a point that lies on all the paths with positive probability (not necessarily with probability one). For $d = 2$, any (finite) number of paths intersect; for $d = 3$, any two paths intersect, but not three; for $d \geq 4$, not even two paths intersect (see Ref. 12). Dual statements for self-intersections of a Brownian path exist (see Ref. [8, thm.21.1, p. 45]).

Oscillations

The dimension of the *graph* of d-dimensional Brownian motion equals $3/2$ (see Ref. 13) a fact that is closely related to the degree of Hölder regularity as indicated by the modulus of continuity (see (2)). For $d \geq 2$, the dimension of the *path* of d-dimensional Brownian motion equals 2, meaning that a Brownian particle moves so erratically that its path almost describes a surface. However, the area of the path of a two-dimensional Wiener process is zero.

Recurrence

Planar Brownian motion is recurrent meaning that it will return infinitely many times arbitrarily close to any given point; however, given any point different from zero it will avoid that point a.s. For $d \geq 3$, Brownian motion is transient, that is, $\lim_{t\to\infty} |W(t)| = \infty$, meaning that it will eventually leave and never return to any given area, no matter where it starts (see Ref. [8, sect. I.18] or Ref. 12).

Using the concept of Green function, involved formulas are available for the expected time that d-dimensional Brownian motion spends in a ball with given center and radius (see Ref. 12).

Reading

Most of the facts presented can be found in the following books, in order of accessible to most technical: Refs [5–8]. A very complete reference on formulas, especially related to stopping times, is Ref. 14. For second-order calculus, see Ref. 10. For stochastic integration with respect to a Wiener process and diffusion properties, see Ref. 9.

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(*See also* **Exceedance over threshold**; **Exceedance probability**; **Extremal events**; **Fractal dimensions**; **Gaussian process**; **Gauss–Markov theorem**; **Lévy processes**; **Long-range dependence**; **Markov process**; **Normal and multinormal distribution**; **Random walk**; **Stochastic process**; **Time series**; **Waiting time**)

RUDOLF H. RIEDI

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